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FOUNDATIONS

Mostowski, A. On a system of axioms which has no recursively enumerable arithmetic model. Fund. Math. 40, 56–61 (1953).

Let A be the conjunction of the axioms of a consistent finitely axiomatized system s and let $R_1^{(e_1)}, \cdots, R_p^{(e_p)}$ be the predicates which occur in A, where the superscripts denote the number of arguments. Then there are relations $\mathrm{R}_1^{(e_1)}, \cdots, \mathrm{R}_p^{(e_p)}$ over the positive integers which satisfy A. Such an ordered p-tuple of relations is called an arithmetic model of S. An arithmetic model is said to be of class P_n (or Q_n), if $\mathrm{R}_j^{(e_j)} \varepsilon P_n^{(e_j)}$ (or $\varepsilon Q^{(e_j)}$), for $j=1, \cdots, p$, where the classes $P_n^{(k)}$, $Q_n^{(k)}$ are defined as follows: if $\mathfrak{n}$ is a k-triple of positive integers, then a set $A \subset I^k$ is said to belong to $P_0^{(k)}$ if there is a recursive predicate $T^{(k)}$ such that $\mathfrak{n} \varepsilon A \equiv T^{(k)}(\mathfrak{n})$; we set $Q_0^{(k)} = P_0^{(k)}$; for $n \geqq 1$, A belongs to $P_n^{(k)}$ (or $Q_n^{(k)}$) if there is a string of n quantifiers, alternately universal and existential, and a recursive predicate $T^{(n+k)}$, such that $\mathfrak{n} \varepsilon A \equiv (\exists x_1)(x_2) \cdots T^{(n+k)}(\mathfrak{n}, x_1, \cdots, x_k)$ (or $\mathfrak{n} \varepsilon A \equiv (x_1)(\exists x_2) \cdots T^{(n+k)}(\mathfrak{n}, x_1, \cdots, x_k)$). Kleene has shown that every consistent finitely axiomatizable system has a model of class $P_2 \cap Q_2$ [Introduction to metamathematics, Van Nostrand, New York, 1952, p. 394; these Rev. 14, 525]. Mostowski here constructs a finitely axiomatized system S which has no model of class P_1. The required system is a modification of the axioms for set-theory as formulated by Bernays. *I. Novak Gál.*

Mostowski, A. A lemma concerning recursive functions and its applications. Bull. Acad. Polon. Sci. Cl. III. 1, 277–280 (1953).

Lemma: Zu jeder rekursiven Funktion $F(n)$ gibt es eine primitiv-rekursive Funktion $H(n)$, die monoton gegen ∞ wächst, so dass $F(H(n))$ primitiv-rekursiv ist. Als Anwendung ergibt sich, dass es zu jeder reellen Zahl $a = \sum_{n=1}^{\infty} W(n)/2^n$ mit $W(n)=0$ oder $W(n)=1$ und rekursivem $W(n)$ primitiv-rekursive Funktionen $M(n)$, $N(n) \neq 0$, $P(n) \neq 0$ mit $\lim P(n) = \infty$ und $|a - M(n)/N(n)| < 1/P(n)$ gibt. Es wird auch noch eine elementare $\forall\exists$-Menge definiert, die keine elementare $\exists$-Menge ist. *P. Lorenzen.*

*Myhill, J. R. **Three contributions to recursive function theory.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 50–59. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

This deals with Gödel's theorem, Hilbert's tenth problem on Diophantine equations, and a problem of Post on undecidable sets. Using ideas of Post and Rosenbloom, the author gives a greatly simplified version of Gödel's incompleteness theorem which entirely avoids arithmetization. Like Post, he deals with systems of strings of symbols, with one axiom, and with rules of procedure of a simple form called normal. Instead of the Gödel enumeration, he uses the notion of the representation of one system in another.

Hilbert's tenth problem, which is still unsolved, is to find a decision procedure for equations obtained by equating two polynomials in several integer variables. M. Davis and L. Kálmár have proved the undecidability of more complicated problems. Kálmár showed the undecidability of the problem whether a given elementary function vanishes identically. The author shows the undecidability of the problem whether $f(v)$ vanishes identically, where $f(v)$ is the minimum for $x_1, \cdots, x_n < v$, of a polynomial $\phi(x_1, \cdots, x_n, v)$. This is more like Hilbert's problem than those of Davis and Kálmár, since it deals with polynomials.

In connection with Post's problem of the construction of undecidable recursively enumerable sets not of maximum degree of undecidability, the author discusses the undecidable sets called "simple" and "creative" by Post, and those called "pseudosimple" by J. Dekker. He introduces the notions of a "pseudocreative" set, and of a set creative with respect to an effective operator. Whether any of these sets provides an answer to Post's problem is left undecided. *O. Frink* (State College, Pa.).

Grzegorczyk, Andrzej. Some classes of recursive functions. Rozprawy Mat. 4, 46 pp. (1953).

A class $\mathfrak{X}$ of functions over the non-negative integers is closed under limited recursion if it satisfies the following condition: If g, h, j are functions of $\mathfrak{X}$, and if f satisfies (a) $f(\mathfrak{u}, 0) = g(\mathfrak{u})$, (b) $f(\mathfrak{u}, x+1) = h(\mathfrak{u}, x, f(\mathfrak{u}, x))$, (c) $f(\mathfrak{u}, x) \leqq j(\mathfrak{u}, x)$, where $\mathfrak{u}$ is a sequence of integers, then f also belongs to x. Now consider the following sequence of recursive functions:

$$f_0(x, y) = y+1, \quad f_1(x, y) = x+y, \quad f_2(x, y) = (x+1)(y+1),$$

and for $n \geqq 2$

$$f_{n+1}(0, y) = f_n(y+1, y+1),$$
$$f_{n+1}(x+1, y) = f_{n+1}(x, f_{n+1}(x, y)).$$

Then $\mathcal{E}^n$ is defined to be the smallest class of functions which includes $S(x) = x+1$, $U_1(x, y) = x$, $U_2(x, y) = y$, $f_n(x, y)$ and is closed under the operations of substitution, identification of variables, substitution of a constant and limited recursion. It is shown that $\mathcal{E}^0 \subset \mathcal{E}^1 \subset \cdots \subset \mathcal{E}^n \subset \mathcal{E}^{n+1} \subset \cdots$ and that $\bigcup_{n=1}^{\infty} \mathcal{E}^n$ is the class of primitive recursive functions. It turns out that every general recursive function $f(\mathfrak{u})$ can be represented in the form $f(\mathfrak{u}) = A(\mu x(B(\mathfrak{u}, x) = 0))$, where A, B are functions of the class $\mathcal{E}^0$. Also that every recursively enumerable set can be enumerated by a function of $\mathcal{E}^0$, in spite of the fact that for each $f \varepsilon \mathcal{E}^0$ there is an integer K_0 such that for every n, $f(n) < n + K_0$. It is shown that $\mathcal{E}^3$ is the class of elementary functions of Kálmár. Various definitions of $\mathcal{E}^3$ are given and shown to be equivalent. There are also discussions of a few related questions. *I. Novak Gál* (Ithaca, N. Y.).

Skolem, Th. Some considerations concerning recursive functions. Math. Scand. 1, 213–221 (1953).

This contains six different observations concerning facts of recursive arithmetic. The first contribution gives a simple proof of an undecidability theorem. A class of numbers is

defined, and it is shown that there is no decision procedure to determine whether a number belongs to the class or not. Next, the author shows how the values of the variables satisfying a general recursive relation can be given as primitive recursive functions of a single parameter. The third item gives a method of replacing general recursive functions by primitive recursive functions.

In connection with Gödel's first theorem of non-deducibility, it is pointed out that we may have primitive recursive functions $f(x)$ and $F(x)$ with different definitions, and such that $f(n)=F(n)$ is provable for each individual n; nevertheless, the identity $f(x)\equiv 0$ may be provable, while $F(x)\equiv 0$ is not. In the fifth section the notion of a function of large oscillation is used to generalize a previous result of the author giving a canonical form for recursive functions. Finally, using the same device, he derives a new normal form for general recursive functions of one variable. The emphasis in these remarks is on simplicity of proof. *O. Frink.*

Goodstein, R. L. Permutation in recursive arithmetic. Math. Scand. **1**, 222–226 (1953).

Since the recursive definition of addition of natural numbers is asymmetrical, the commutative property of addition requires proof. Inductive proofs of commutativity of the sum of a specified number of terms are readily given, but an independent proof is required for the case of a finite, but unspecified, number of terms. The author supplies such a proof, after defining recursively the result of a sequence of transpositions. *O. Frink* (State College, Pa.).

***Bar-Hillel, Yehoshua. On recursive definitions in empirical sciences.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 160–165. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

The author points out that recursive definitions are not definitions in the ordinary sense. They involve circularity, but only in appearance, not in reality. Recursive definitions are unavoidable in the empirical sciences. Instead of giving illustrations of this for the physical sciences, as is usually done, the author uses examples from the science of linguistics. *O. Frink* (State College, Pa.).

Kreisel, G. On a problem of Henkin's. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 405–406 (1953).

Let Σ be any standard formal system adequate for recursive number theory. A formula (having some integer q as its Gödel number) can be found which expresses the proposition that the formula with Gödel number q is provable in Σ. The author shows that depending on the formula used to express the notion of provability in Σ, the required formula may be provable in Σ or independent in Σ. *I. Novak Gál* (Ithaca, N. Y.).

***Kreisel, G. Note on arithmetic models for consistent formulae of the predicate calculus. II.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 39–49. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

It was previously shown [Fund. Math. **37**, 265–285 (1950); these Rev. **12**, 790] that any consistent formula of the extended predicate calculus (which contains both predicate symbols and function symbols) has a model in which the function symbols are replaced by primitive recursive functions and the predicate symbols are replaced by predicates of $Z\mu$ [cf. Hilbert and Bernays, Grundlagen der Mathematik, Bd. II, Springer, Berlin, 1939, p. 293] with just 2 quantifiers. The author now exhibits a formula S of the extended predicate calculus which has no model in which the function symbols are replaced by computable functions and the predicates by predicates of $Z\mu$ with only one quantifier, so that the above result is a best possible. *I. Novak Gál* (Ithaca, N. Y.).

Hasenjaeger, Gisbert. Topologische Untersuchungen zur Semantik und Syntax eines erweiterten Prädikatenkalküls. Arch. Math. Logik Grundlagenforsch. **1**, 99–129 (1952).

This paper contains detailed topological proofs of the completeness theorem and the Skolem-Löwenheim theorem for the first-order predicate calculus with and without identity. The methods are similar to those of Rasiowa and Sikorski [Fund. Math. **37**, 193–200 (1950); these Rev. **12**, 661] and Beth [Indagationes Math. **13**, 436–444 (1951); these Rev. **13**, 614], but were developed independently. *I. Novak Gál* (Ithaca, N. Y.).

Shoenfield, Joseph R. A relative consistency proof. J. Symbolic Logic **19**, 21–28 (1954).

Let C be an axiom system formalized within the first order predicate calculus and let C' be a predicative extension of C, so that for each predicate at C there is a corresponding class in C'. The author gives a new and very natural proof of the following theorems: (i) C' is consistent if and only if C is consistent; (ii) every theorem of C' which can be stated in C is a theorem of C. His proof is based on a variant of the first ϵ-theorem of Hilbert and Bernays [Grundlagen der Mathematik, vol. 2, Springer, Berlin, 1939, p. 30]. It is an improvement over the previous proofs in that it provides a direct method for obtaining a contradiction in C once we have a proof of contradiction in C'. *I. Novak Gál.*

Rose, Alan. Fragments of the m-valued propositional caclulus. Math. Z. **59**, 206–210 (1953).

L. Henkin showed that any fragment of the 2-valued propositional calculus in which material implication is definable, can be formalized. Extending this result, the author gives a formalization for any fragment of the m-valued calculus with one designated truth value, in which the implication function of Post and Łukasiewicz is definable. He does this by means of four rather complicated axiom schemes, with two rules of procedure. It is proved that this formalization is weakly complete, and that if the axiom schemes are replaced by the corresponding axioms and a substitution rule is added, the resulting formalization is strongly complete. *O. Frink* (State College, Pa.).

Rasiowa, H., and Sikorski, R. On satisfiability and decidability in non-classical functional calculi. Bull. Acad. Polon. Sci. Cl. III. **1**, 229–231 (1953).

Rasiowa, H., and Sikorski, R. Algebraic treatment of the notion of satisfiability. Fund. Math. **40**, 62–95 (1953).

Let S be a consistent system of sentential calculus containing at least t, $\cdot$, $\rightarrow$, possibly other sentential operators. The rules of inference are modus ponens and substitution and all theorems of the positive sentential calculus are theorems of S. Then S determines a corresponding system S^* of functional calculus with individual variables $x_1, x_2, \cdots$, k-argument functional variables F_1^k, F_2^k, $\cdots$ and quantifiers

$\sum_{x_i}$, $\prod_{x_j}$, with the usual rules of inference. The theorems of S^* are all substitutions of theorems of S and their consequences. S determines an abstract algebra (S-algebra) with algebraic operations corresponding to the logical operations and with unit element e. If an S-algebra is a complete lattice, it is called an S^*-algebra. It is assumed that S-algebras A have the property: given an enumerable set of infinite sums and products in A, $a_n = \sum_i a_{n,i}$, $b_n = \prod_i b_{n,i}$, then there is an isomorphism of A into an S^*-algebra which preserves these sums and products. Let J be a non-empty set, A an S^*-algebra. Each formula $\alpha \in S^*$ can be interpreted as an algebraic functional $(J, A)\phi_\alpha$. A set $R \subset S^*$ is said to be satisfiable in $J \neq 0$, if there is an S^*-algebra A such that all the functionals $(J, A)\phi_\alpha$ $(\alpha \in R)$ assume the value $e \in A$ for a common (independent of α) substitution for the variables x_i, $F_m{}^k$. The set R is satisfiable if it is satisfiable in a set $J \neq 0$. A formula $\alpha \in S^*$ is valid if $(J, A)\phi_\alpha = e \in A$ for every $J \neq 0$ and every S^*-algebra A. Let I be the set of positive integers. S_k denotes the classical, S_λ the Lewis S_4, S_χ the Heyting, S_π the positive sentential calculus. The first of these two papers announces the following results. Theorem I. A formula α is provable in S^* if and only if α is valid (or: if α is valid in the set I, i.e. if $(I, A)\Phi = e \in A$ identically for each S^*-algebra A). Each consistent set $R \subset S^*$ is satisfiable in the set I. Theorem II. If the system S contains the negation sign and if the formula $(-(\sigma \to \sigma)) \to \tau$ is a theorem in S, then each satisfiable set $R \subset S^*$ having the deduction property is consistent and satisfiable in the set I.

If X is a topological space then $C(X)$ $(H(X))$ denotes the class of all subsets (of all open subsets) of X. Then $C(X)$ is an $S_\lambda{}^*$-algebra (with the usual operations $a+b$, $a \cdot b$, and $a \to b = (X-a)+b$, $-a = X-a$, $\Box a =$ the interior of a), and $H(X)$ is an $S_\chi{}^*$-algebra and $S_\pi{}^*$-algebra (with the usual operations $a+b$, $a \cdot b$, and $a \to b =$ the interior of $((X-a)+b)$ $-a = a \to 0$).

Theorem III. There is a topological space X such that (i) a formula $\alpha \in S_\lambda{}^*$ $(S_\chi{}^*, S_\pi{}^*)$ is provable if and only if $(I, C(X))\Phi_\alpha = X$ $((I, H(X))\Phi_\alpha = X)$ identically, (ii) a set $R \subset S_\lambda{}^*$ $(S_\chi{}^*, S_\pi{}^*)$ having the deduction property is satisfiable if and only if it is satisfiable in I and in $C(X)$ $(H(X))$. From Theorem III (i) follows Theorem IV: A formula $\alpha \in S_\pi{}^*$ $(S_\chi{}^*)$ is provable in $S_\pi{}^*$ $(S_\chi{}^*)$ if and only if it is provable in $S_\chi{}^*$ (in $S_\lambda{}^*$ provided that the expressions $\beta \to$, $-\beta$, $\prod_{x_i} \beta$ of $S_\chi{}^*$ are interpreted in $S_\lambda{}^*$ as $\Box(\beta \to)$, $\Box(-\beta)$, $\Box \prod_{x_i} \beta$ respectively). Theorem V. Let $\alpha, \beta \in S_\lambda{}^*$ $(S_\chi{}^*, S_\pi{}^*)$. If $\Box\alpha + \Box\beta$ $(\alpha+\beta)$ is provable in $S_\lambda{}^*$ (in $S_\chi{}^*$, in $S_\pi{}^*$), then either α or β is provable in $S_\lambda{}^*(S_\chi{}^*, S_\pi{}^*)$. If $\sum_{x_i} \Box\alpha$ $(\sum_{x_i} \alpha)$ is provable in $S_\lambda{}^*$ $(S_\chi{}^*, S_\pi{}^*)$, then there is an integer q such that the substitution $\alpha\binom{x_q}{x_i}$ is provable in $S_\lambda{}^*(S_\chi{}^*, S_\pi{}^*)$. Theorem VI. Each formula $\beta \in S_\lambda{}^*$ $(S_\chi{}^*, S_\pi{}^*)$ of the form $\beta = \Xi\alpha$, where α contains no quantifier and Ξ is a sequence of the signs $\sum_{x_i} \Box$, $\prod_{x_j}$ (of the signs $\sum_{x_i}$ and $\prod_{x_j}$), is decidable. Similar theorems also hold for the minimal functional calculus $S_\mu{}^*$ determined by the minimal sentential calculus S_μ.

The second paper gives detailed proofs of Theorems I–IV and a few related results. The methods used are generalizations of the topological methods first used by the authors to give a proof of the Gödel completeness theorem.

I. Novak Gál (Ithaca, N. Y.).

Moh, Shaw-Kwei. Logical paradoxes for many-valued systems. J. Symbolic Logic 19, 37–40 (1954).

The author first defines an implication function Cpq to be a function such that the rule "From P and CPQ we can infer Q" is valid. He then defines $(Cp)^i q$ by "$(Cp)^1 q$ is to be Cpq, $(Cp)^{i+1} q$ is to be $Cp(Cp)^i q$" and shows that if in a system we can assert the proposition Cpp and the rule "If $(Cp)^{n+1} q$ then $(Cp)^n q$" then the system is led into an inconsistency by the class $\lambda x.(C \in xx)^n p$. He then proves a number of theorems concerning the validity of the above rule of absorption in various systems. Finally he raises some objections to Łukasiewicz's interpretation of 3-valued logic and suggests an alternative interpretation. *A. Rose.*

Martin, Norman M. The Sheffer functions of 3-valued logic. J. Symbolic Logic 19, 45–51 (1954).

The author first obtains a number of necessary conditions for a function to be a Sheffer function of m-valued logic. He then isolates all Sheffer functions of 3-valued logic and shows that there are 3774 such functions. *A. Rose.*

Please see the note on p. 1140

Janiczak, A. Undecidability of some simple formalized theories. Fund. Math. 40, 131–139 (1953).

This paper is a contribution to the topics expounded in Tarski's "Undecidable theories" [North-Holland Publ. Co., Amsterdam, 1953; these Rev. 15, 384]. Since the essentially undecidable quantification theory (q.t.) of non-densely ordered rings can be weakly interpreted in the q.t. of two equivalence relations R_0, R_1, their q.t. is undecidable; analysis of the proof shows that it remains undecidable even if the common part of R_0 and R_1 is the identity relation, the equivalence classes of R_1 are all pairs, those of R_0 consist either of 1, 5, 6 or infinitely many elements. The result is optimal because the author provides a decision method for the q.t. of a single equivalence relation. By simple modifications of his main construction he establishes the undecidability of (a) the third-order monadic predicate calculus, which was proved independently by Church and Quine [J. Symbolic Logic 17, 179–187 (1952), Theorem IV, p. 185; these Rev. 14, 233], (b) the q.t. of one equivalence relation and one one-one relation, (c) the q.t. of one function and one one-one relation. Possible improvements of these results are suggested by the author's questions whether the q.t. of (i) a single function, (ii) k one-one relations is decidable. Reviewer's note. The q.t. of a single one-one relation R, where $(x)\{R[x, Fx] \& R[F^{-1}x, x]\}$, is decidable. For, since $F^p x = x \& F^q x = x . \leftrightarrow . F^{(p,q)} x = x$, (p, q) being the h.c.f. of p and q, and since

$$F^p x = x \& F^q x \neq x . \leftrightarrow . (F^{p_1} x = x \& F^{i_1} x \neq x) \vee \cdots \vee (F^{p_k} x = x \& F^{i_k} x \neq x),$$

$0 < i_j < p_j$, where p_j are the factors of p which do not divide q, the standard elimination of quantifiers (s.e.q.) leads to a formula of the propositional calculus made up of the following propositions: there are at least n individuals x (i) in the universe, (ii) on a k-cycle, $F^i x \neq x$ for $0 < i < k$, $F^k x = x$, (iii) not on a $k_1, \cdots, k_m$ cycle. Since such a formula can be satisfied in a specifiable finite universe if it can be satisfied at all, the theory is decidable.

The author also asks whether the q.t. of an ordering relation [R transitive, and $\neg R(x, y) \leftrightarrow . x = y \vee R(y, x)$] is decidable. To apply s.e.q., introduce three sets of new non-logical constants: there are at least n individuals (i) 'greater' than x: $M_n(x)$, (ii) less than x: $m_n(x)$, (iii) greater than x and less than y: $B_n(x, y)$. If A is made up of these constants, R, $=$, &, $\neg$, and if N is the largest number of individuals

mentioned in A, order the variables in A (other than x) in all possible ways with all possible spacings $\leq N+1$ between them; for each such arrangement $(Ex)A$ is decidable. Since $R(x,x)$, $B_n(x,x)$ are false, and since the existence of at least n individuals is equivalent to $(Ex)M_{n-1}(x)$, the s.e.q. leads finally to a formula with the monadic predicates $M_i(x)$, $m_i(x)$, $i \leq N_1$: such a formula is valid precisely when it is derivable from the formulae $M_{i+1}(x) \to M_i(x)$, $m_{i+1}(x) \to m_i(x)$ with $i < N_1$, and this is decidable since the monadic predicate calculus (of first order) is decidable. *G. Kreisel.*

Stanley, Robert L. Note on a paradox. J. Symbolic Logic 18, 233 (1953).

The author constructs paradoxical sets whose complements are also paradoxical. One such set, analogous to the Russell set of all sets not members of themselves, is the set of all sets x such that the complement of x is not a member of x. *O. Frink* (State College, Pa.).

Wang, Hao. Between number theory and set theory. Math. Ann. 126, 385–409 (1953).

Let G be the system of set theory based on the axiom of extensionality, the Aussonderungsaxiom and the axiom of finite sets $[(x)(y)(\exists z(W)(W \varepsilon z \to . W \varepsilon x \vee W=y))]$ and let G' be Zermelo set theory without the axiom of infinity. G (also G') and number theory are mutually interpretable and G and G' have a simple common denumerable model (this model is due to Ackermann and uses the integers with the following membership relation: $m \varepsilon^* n$ if and only if $[n/2^m]$ is odd). A predicate E can be constructed in G which enumerates the sets of the model. Let G^* be the system obtained by adding to G as a new axiom: For every set x there is a natural number m such that x is $E(m)$. Then G^* turns out to be stronger than G' and any 2 models for G^* are isomorphic whenever their parts for number theory in G^* are isomorphic. There is a discussion of predicative extensions P' of G' which contains a class (collection of sets) for each predicate of G' and also of the role of induction in such a predicative extension of G. *I. Novak Gál.*

Kreisel, G. A variant to Hilbert's theory of the foundations of arithmetic. British J. Philos. Sci. 4, 107–129 (1953); errata and corrigenda, 357 (1954).

The author here concerns himself with the problem of interpreting classical arithmetic using quantifier-free methods. A quantifier-free system of arithmetic is based on the following axioms and schemata ("$<\cdot$" denotes a well-ordering relation):

$$a+0=a, \quad a+b'=(a+b)', \quad a\cdot 0=0, \quad a\cdot b'=(a\cdot b)+a,$$

$$\frac{A(0),\, A(u) \to A(u')}{A(u)},$$

$$\frac{A(0) \& . (f(u) <\cdot u \vee u=0) . \& . A(f(u)) \to A(u)}{A(u)}$$

He also extends his methods to parts of analysis and points out the difficulties encountered. He ends with a sequence of notes giving technical details of his presentation.

I. Novak Gál (Ithaca, N. Y.).

Brouwer, L. E. J. Addenda and corrigenda on the role of the principium tertii exclusi in mathematics. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 104–105 (1954). (Dutch)

Additions and corrections to a paper published in Dutch [Wis- en Natuurk. Tijdschr. 2, 1–7 (1923)] and in German [J. Reine Angew. Math. 154, 1–7 (1924)]. Some are terminological; the others concern questions treated in other papers by the author [Verh. Akad. Wetensch. Amsterdam. Sect. 1. 13, no. 2 (1923); Akad. Wetensch. Amsterdam, Proc. 29, 866–867 (1926); 31, 374–379 (1928)].

A. Heyting (Amsterdam).

Finsler, P. Die Unendlichkeit der Zahlenreihe. Elemente der Math. 9, 29–35 (1954).

Completely disregarding the wealth of results and methods of mathematical logic, the author proposes the following principles for set theory: (a) every set determines its elements; (b) if possible, two sets M and N are identical; (c) if possible, M is a set. From his comments it appears that the author would adopt all the operations of general set theory [e.g., Ackermann, Math. Ann. 114, 305–315 (1937)]; "possible" would naturally be taken to mean: consistent over a suitable logic, so that, according to (b), $M=N$ is to be added as an axiom if it is consistent. Reviewer's note: in special cases, e.g. if M and N are decidable sets of integers, the author's principle (b) is quite natural since, if $M=N$ is consistent, $M \neq N$ is ω-inconsistent; but for general M and N, $M=M'$, $N=N'$ may separately be consistent, though $M'=N'$ is refutable. *G. Kreisel* (Reading).

Grünbaum, Adolf. Whitehead's method of extensive abstraction. British J. Philos. Sci. 4, 215–226 (1953).

Whitehead's Method was meant to 'describe what a point is' by showing 'how the geometric relations between points issue from the ultimate relations between the ultimate things which are the immediate objects of knowledge'. In accordance with this program, points should satisfy two conditions, stated by C. D. Broad as follows: (1) points must have to each other the kind of relations which geometry demands; and (2) points must have to finite areas and volumes such a relation that a reasonable sense can be given to the statement that such areas and volumes can be exhaustively analysed into sets of points.

The author sets out to show, against Ushenko's recent defence of it [Schilpp, Albert Einstein: philosopher-scientist, Library of Living Philosophers, Evanston, Ill., 1949, pp. 607–645; these Rev. 11, 707], that the Method, and all similar positivistic constructive attempts, must fail; his arguments are that (i) even if the existence of a denumerable actual infinite is somehow certifiable by sense awareness, sense awareness cannot suggest the idea of a super-denumerable collection of perceptible regions, which is needed in order to avoid Zeno's paradox of plurality, (ii) that the convergence of Whitehead's abstractive classes is ambiguous, and (iii) that these classes do not belong to the domain of sense awareness.

Though the author's conclusions are probably correct, argument (i) does not seem fully convincing on account of the Skolem-Löwenheim paradox.

E. W. Beth.

★Riabouchinsky, Dimitri. La définition des nombres par leur valeur numérique et par leur origine; rôle de ce concept en philosophie mathématique. Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 208–214. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

ALGEBRA

Roy, Purnendu Mohon. A note on the unreduced balanced incomplete block designs. Sankhyā 13, 11–16 (1953).

A balanced incomplete block design consisting of all combinations of k out of v elements is called unreduced if $(b, r, \lambda)=1$. The author shows that the designs with $k=2$ or $v-2$ or $v-1$ and the design with $v=8$ and $k=5$ or 3 are the only unreduced designs. Further properties of these designs are investigated. *H. B. Mann* (Columbus, Ohio).

Paige, Lowell J., and Wexler, Charles. A canonical form for incidence matrices of finite projective planes and their associated Latin squares. Portugaliae Math. 12, 105–112 (1953).

The incidence matrix for a projective plane with n^2+n+1 points may be normalized by being broken into blocks where the rows of any one block are the lines through a point and the columns are the points on a line. The top and left-hand blocks may be written out explicitly and any other block is an n by n permutation matrix. This normalized form is directly related to the well known fact that a complete set of $n-1$ mutually orthogonal n by n latin squares determines a plane. *Marshall Hall, Jr.* (Columbus, Ohio).

Tenca, Luigi. Minori circolanti contenuti in un dato determinante circolante. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 589–598 (1950).

A circulant determinant is a determinant of the form

$$D=\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \cdots & & & \\ a_n & a_1 & \cdots & a_{n-1} \end{vmatrix}.$$

This paper investigates the minors of a given circulant determinant which are also circulants. It investigates in particular the cases in which $a_1, a_2, \cdots, a_n$ are (a) the nth roots of unity, (b) the elements of an arithmetic progression, and (c) the elements of a geometric progression. *G. B. Price* (Lawrence, Kansas).

Tenca, L. Relazioni fra determinanti ricavati da una particolare matrice. Period. Mat. (4) 30, 211–215 (1952).

The author evaluates determinants whose elements are taken from the matrix

$$\|a^m\ (a+d)^m\ (a+2d)^m \cdots (a+nd)^m \cdots\|, \qquad m=\cdots, -2, -1, 0, 1, 2, \cdots,$$

where a and d are real numbers. For example, the author evaluates

$$D=|(a+r_md)^m\ (a+[r_m+s_m]d)^m\ (a+[r_m+2s_m]d)^m \cdots (a+[r_m+\{n-1\}s_m]d)^m|, \qquad m=0, 1, 2, \cdots, n-1,$$

where $r_0, r_1, \cdots, r_{n-1}, s_0, s_1, \cdots, s_{n-1}$ are any positive integers, and r_0 and s_0 may be zero also. The author also establishes properties of sequences of determinants whose elements are taken from the given matrix. *G. B. Price.*

Vil'ner, I. A. On relations among the minors of certain matrices. Uspehi Matem. Nauk (N.S.) 8, no. 5(57), 139–146 (1953). (Russian)

This paper contains theorems which correspond to the theorem on "development by non-conforming cofactors". One such theorem is the following, which concerns minors of an $(n+1)\times n$ matrix. Let t_i, j_i be integers, $1\leqq t_1<\cdots<t_{k+1}\leqq n$; $0\leqq j_1<\cdots<j_k<n$. Let M_0 be the $n\times n$ minor determinant on the first n rows; let M_0' be the minor determinant obtained by striking out rows with indices t_i (all i) and columns with indices j_i+1 (all i). Let M_s be the $n\times n$ minor determinant on all rows except the t_sth; let M_s' be the minor determinant obtained by striking out the $(n+1)$th row and rows with indices t_i $(i\neq s)$ and columns with indices j_i+1 (all i); then the relation $M_0M_0'+\sum_{s=1}^{k+1}M_sM_s'=0$ holds. More complicated theorems are given for minors of an $(n+r)\times n$ matrix. *J. L. Brenner* (Pullman, Wash.).

Egerváry, E. On a lemma of Stieltjes on matrices. Acta Sci. Math. Szeged 15, 99–103 (1954).

If all n of a chain of principal minors of a real matrix $A=(a_{ij})$ are positive, and $a_{ij}<0$ for $i\neq j$, then

$$A=(1-M)Q(1-N),$$

where $m_{ij}=0$ if $i\leqq j$, $m_{ij}>0$ if $i>j$, $n_{ij}=0$ if $i\geqq j$, $n_{ij}>0$ if $i<j$ and $Q=\|\text{diag}\,(q_1, \cdots, q_n)\|$ with $q_i>0$. Hence, $A^{-1}=(1+N+\cdots+N^{n-1})Q^{-1}(1+M+\cdots+M^{n-1})$ and the elements of A^{-1} are all positive. That A^{-1} has positive elements is also proved by a different method when $a_{ij}<0$ for $i\neq j$ is weakened to $a_{ij}\leqq 0$ for $i\neq j$ and at least one $a_{ij}<0$ both for $i<j$ and $i>j$, $j=1, 2, \cdots, n$. *W. Givens.*

Hsu, P. L. On symmetric, orthogonal, and skew-symmetric matrices. Proc. Edinburgh Math. Soc. (2) 10, 37–44 (1953).

Let Σ (resp. K) be the set of real symmetric (resp. skew-symmetric) $p\times p$ matrices. Using the Cayley parametrization connecting K and the set of orthogonal matrices, the author shows that every S in Σ is representable as

$$S=[2(I+X)^{-1}-I]D_\theta[2(I+X)^{-1}-I]'$$

for some X in K, where D_θ is a diagonal matrix whose diagonal elements are the characteristic roots of S. Upon removing from Σ a set of measure zero, and imposing some simple conditions on $(I+X)^{-1}$, X is uniquely determined by S. If $f(S)$ is a scalar function of S which depends only upon the characteristic roots of S, the author uses the above transformation to express $\int_\Sigma f(S)\,dv$ as an integral over a portion of the space of characteristic roots. This last result is applied to the question of probability distribution of characteristic roots. *I. Reiner* (Princeton, N. J.).

Ansari, A. R., and Shah, S. M. A note on certain nilpotent matrices. Math. Student 20 (1952), 113–114 (1953).

This paper contains some elementary theorems on nilpotent matrices. For example, it is shown that if $B^{(n)}$ is nilpotent, then $f(1)$ is an n-fold characteristic root of $f(I+B)$. *I. Reiner* (Princeton, N. J.).

Egan, M. F., and Ingram, R. E. On commutative matrices. Math. Gaz. 37, 107–110 (1953).

In a recent paper by Drazin, Dungey, and Gruenberg [J. London Math. Soc. 26, 221–228 (1951); these Rev. 12, 793], simpler proofs were given of some theorems of McCoy [Bull. Amer. Math. Soc. 42, 592–600 (1936); Trans. Amer. Math. Soc. 36, 327–340 (1934)] on commutative matrices. The present paper further simplifies the proofs of some of the results. *I. Reiner* (Princeton, N. J.).

Eaves, J. C. On sets of matrices having a delayed commutativity property. J. Elisha Mitchell Sci. Soc. **68**, 46–54 (1952).

Let $A, B, C, \cdots$ denote $n\times n$ square matrices with complex elements and with characteristic roots $\alpha_1, \alpha_2, \cdots, \alpha_n$ of A, $\beta_1, \beta_2, \cdots, \beta_n$ of B, $\gamma_1, \gamma_2, \cdots, \gamma_n$ of $C, \cdots$. The author proves that certain finite sets of matrices $S=\{A, B, C, \cdots\}$ have the "p" property and the triangle property. The matrices of S are said to have the "p" property if the $\alpha_i, \beta_i, \gamma_i, \cdots$ $(i=1, 2, \cdots, n)$ can be ordered such that for any rational function $f(x, y, z, \cdots)$ the characteristic roots of the matrix $f(A, B, C, \cdots)$ are given by $f(\alpha_i, \beta_i, \gamma_i, \cdots)$ $(i=1, 2, \cdots, n)$. The matrices of S are said to have the triangle property if there exists a unitary matrix U such that U^*AU, U^*BU, $U^*CU, \cdots$ are all triangular matrices. All results are established by the use of invariant vector spaces. The author first proves by this method the known results that, if $R=\{A_1, A_2, \cdots, A_s\}$ is a finite set of $n\times n$ commutative matrices, then the matrices of R have the triangle property and the "p" property. Let $[A,B]=AB-BA$; then A and B are called quasi-commutative if $[A, B]=C$, where C commutes with both A and B. By the use of properties of common invariant vector spaces, the author proves that two quasi-commutative matrices A and B have the triangle property and the "p" property. This result is next extended to semi-quasi-commutative matrices. If $[A, B]=C$, $[C, B]=0$, $[C, A]=D$, $[D, A]=0$, then A and B have a delayed commutativity property and are called semi-quasi-commutative matrices. If A and B have the delayed commutativity property that $[A, B]=C$, $[C, A]=Q$, $[C, B]=R$, where both Q and R commute with each of A and B, then A and B are defined to be quasi-2,2-commutative. The author proves that a pair of quasi-2,2-commutative matrices has the triangle property and the "p" property. Finally, the author extends the idea of quasi-commutative pairs of matrices to quasi-commutative sets of matrices and establishes certain theorems for them. *G. B. Price.*

Copping, J. Non-associative rings of infinite matrices. J. London Math. Soc. **29**, 177–183 (1954).

Nonassociative rings of infinite matrices are constructed, some with and some without a unit element, which have the following property. To each ring there corresponds a positive integer n such that if every matrix of the ring is partitioned in the form

$$\begin{pmatrix} \alpha & \delta \\ \beta & \gamma \end{pmatrix}$$

where α is a finite matrix of order $n\times n$ then in any product of matrices in the ring the parts α, β, and γ remain invariant under changes of order of multiplication. In particular, a ring of this type is constructed which contains a matrix M for which $M^2M\neq MM^2$; and the ring whose general element is $c_1A_1+\cdots+c_kA_k$, where the c_j are complex numbers and each A_j is a finite product formed from M only, is a nonassociative ring without a unit element. In addition to providing examples of the breakdown of commutativity in rings of infinite matrices, the author gives another difference between the theory of finite matrices and the theory of infinite matrices by giving an example of an infinite matrix N for which NN^2 exists but N^2N does not exist.

R. P. Agnew (Ithaca, N. Y.).

Takeno, Hyôitirô. A theorem concerning the characteristic equation of the matrix of a tensor of the second order. Tensor (N.S.) **3**, 119–122 (1954).

The Newton identities are used to relate the coefficients of the characteristic equation of a matrix to the traces of its powers. *W. Givens* (New York, N. Y.).

Schneider, Hans. Regions of exclusion for the latent roots of a matrix. Proc. Amer. Math. Soc. **5**, 320–322 (1954).

The theorem of Geršgorin [Izvestiya Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) **1931**, 749–754] and A. Brauer [Duke Math. J. **13**, 387–395 (1946); these Rev. **8**, 192] concerning the n circles which contain the characteristic roots of an $n\times n$ matrix is generalized by using the fact that a matrix remains non-singular if its columns are subjected to a permutation $\mu(i)$. The following result is obtained: the characteristic roots of $A=(a_{ik})$ lie in the union of the n regions $|z-a_{ii}|\leqq\sum_{k\neq i}|a_{ik}|$ for $i=\mu(i)$, and $|z-a_{ii}|\geqq|a_{i\mu(i)}|-\sum_{k\neq i,\mu(i)}|a_{ik}|$ for $i\neq\mu(i)$. The same idea is also applied to a theorem of the reviewer [J. Research Nat. Bur. Standards **46**, 124–125 (1951); these Rev. **13**, 311] and P. Stein [ibid. **48**, 59–60 (1952); these Rev. **13**, 813] concerning multiple characteristic roots.

O. Taussky-Todd (Washington, D. C.).

Roth, William E. On the characteristic polynomial of the product of two matrices. Proc. Amer. Math. Soc. **5**, 1–3 (1954).

Let A_1, A_2 be $n\times n$ matrices; rank $(A_1-A_2)\leqq 1$. Let the characteristic polynomial of A_i be $a_i(t^2)-tb_i(t^2)$ $(i=1, 2)$. Then the characteristic polynomial of A_1A_2 is

$$(-1)^n[a_1(t)a_2(t)-tb_1(t)b_2(t)].$$

J. L. Brenner (Aberdeen, Md.).

Parker, W. V. Matrices and polynomials. Amer. Math. Monthly **61**, 182–183 (1954).

The interrelations between the vector $c=(c_1, c_2, \cdots, c_n)$, the polynomial $c(x)=c_1+c_2x+\cdots+c_nx^{n-1}$ and the companion matrix C of $f(x)=x^n-c(x)$ are exploited in a compact yet explicit way. The main theorems on C are proved in the course of establishing the following two results. I. Let $d(x)$ be the greatest common divisor of $f(x)$ and (any) $k(x)$. If $k(C)$ is nonsingular, $d=1$. If rank $k(C)=r<n$ and $g=(g_1, \cdots, g_r, 1, 0, \cdots, 0)$ is (necessarily unique and) such that the vector $gk(C)=0$, then $d(x)=f(x)/g(x)$. II. If A and B are nonderogatory and can be carried into their rational canonical forms by the same similarity, the only solutions of $AX=XB$ are polynomials in B.

W. Givens (New York, N. Y.).

Scott, W. R. Divisors of zero in polynomial rings. Amer. Math. Monthly **61**, 336 (1954).

Abstract Algebra

Sholander, Marlow. Postulates for Boolean algebras. Canadian J. Math. **5**, 460–464 (1953).

Write

$$(a_1, a_2, \cdots, a_n) \text{ for } a_1+(a_2+(a_3+\cdots+(a_{n-1}+a_n)\cdots)),$$

and consider the following three axioms:

P $(a, b, c, b, c, b, a)=b$;
Q $(a, (cc)a, a(b+c), b(cd))=(ba, (II)a, d(b+b), c(bd))$;
R $(x|(y'|y))''=(a|(b'|c))''$ implies that $x=(b|a)|(c'|a)$.

Closure axioms are assumed, I in axiom Q is a fixed element, and in R the abbreviations $a'=a|a$ and $a''=(a')'$ are used. It is shown that P characterizes Boolean groups, P and Q characterize Boolean rings with an identity, and R characterizes Boolean algebras. *B. Jónsson.*

Montague, Richard, and Tarski, Jan. On Bernstein's self-dual set of postulates for Boolean algebras. Proc. Amer. Math. Soc. **5**, 310–311 (1954).

The axiom system in question was first thought to be irredundant [cf. Bull. Amer. Math. Soc. **22**, 458–459 (1916)] but later an error in one of the independence proofs was noted by R. Russell [cf. B. A. Bernstein, Scripta Math. **16**, 157–160 (1950); these Rev. **12**, 583]. The authors show that one of the postulates (either one of the commutative laws) can in fact be derived from the others. *B. Jónsson.*

Zappa, Guido. La teoria dei reticoli e le sue applicazioni. Univ. e Politecnico Torino. Rend. Sem. Mat. **12**, 5–19 (1953).

Exposition of elementary lattice theory, with many known examples. *P. M. Whitman* (Silver Spring, Md.).

Janoš, Ludvík. Properties of the Zassenhaus refinement. Čehoslovack. Mat. Ž. **3**(78), 159–180 (1953). (Russian. English summary)

[For notation and terminology, see V. Kořínek, Rozpravy II. Třídy České Akad. **59**, no. 23 (1949); these Rev. **12**, 667.] In a lattice, two finite descending chains $\{x_i\}$ and $\{y_i\}$, $i=0, 1, \cdots, n$, with the same endpoints are called lower simply similar (l.s.s.) if for some $f(i)$,

$$x_i/x_{i+1}\gamma y_{f(i)}/y_{f(i)+1}, \quad i=0, 1, \cdots, n-1.$$

Theorem: If the lattice is modular, then the Zassenhaus refinements of two chains are not proper refinements if and only if the given chains are l.s.s. Hence if two chains are l.s.s. then $f(i)$ is unique. Some subsidiary properties of γ, γ, and $\tilde{\gamma}$ are explored, both in general lattices and in modular lattices. Theorem: In a modular lattice, given two chains, their Zassenhaus refinements are the minimal l.s.s. refinements thereof, and are the only l.s.s. refinements which lie in the sublattice generated by the elements of the given chains. Some analogous results are found for groups. In particular, in a group with composition series, Zassenhaus refinements are not proper refinements if and only if normal chains are l.s.s. *P. M. Whitman* (Silver Spring, Md.).

Vitner, Čestmír. The semimodular conditions in the lattices. Čehoslovack. Mat. Ž. **3**(78), 265–282 (1953). (Russian. English summary)

[See the preceding review and the reference cited there.] In a lattice S, denote by π_1 the condition that $a/b\gamma c/d$ and $\{a_i\}$ is a maximal chain between a and b together imply $K_d=0$ or K_d has a greatest element, where K_d is the set of a_i for which $a_i \geqq c$ does not imply $a_i\cap(c\cup b)=(a_i\cap c)\cup b$; π_1 is equivalent to lower semimodularity if all chains in S are finite. Define π_2 dually. Then π_1 implies that the lower prime quotient condition is satisfied; the converse holds if all chains of S are finite; and S is modular if and only if π_1 and π_2 hold. Several related results and independence proofs are given. The relationship of the above ideas to lower simple similarity of chains is considered. Theorem: In S, if any two finite chains with the same ends have refinements which are l.s.s., then if $a/b\gamma c/d$, $\{a_i\}$ is a maximal chain between a and b, and c_i denotes $a_i\cap c$, then $\{c_i\}$ is a maximal chain between c and d. *P. M. Whitman* (Silver Spring, Md.).

Curzio, Mario. Su di un particolare isomorfismo di struttura. Ricerche Mat. **2** (1953), 288–300 (1954).

Given a finite group G, denote by $L(G)$ the lattice of its subgroups, and by $S(G)$ the lattice of all cosets of all subgroups of G ordered by set inclusion. An example proves that lattice isomorphism of $S(G)$ and $S(G')$ does not imply ordinary isomorphism of G and G', answering Problem 42 of G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; these Rev. **10**, 673]. However, ordinary isomorphism is implied if G is cyclic, the quaternions, a generalized quaternion group, or an Abelian group of type $(1, 1, \cdots, 1)$. Although $L(G)$ is a sublattice of $S(G)$, an isomorphism ω of $S(G)$ and $S(G')$ need not carry $L(G)$ into $L(G')$; but in each such case there must exist another isomorphism of $S(G)$ and $S(G')$ which does carry $L(G)$ into $L(G')$, and G has the same number of elements as G'. If ω carries normal subgroups into normal subgroups, the centers of G and G' are isomorphic; if also G is Abelian or Hamiltonian then G and G' are isomorphic. Normalizers and the like are also considered. There are troublesome misprints; e.g. in the first line of §1, for "$S(G)$ dei sottogruppi" read "$S(G)$ dei laterali dei sottogruppi". *P. M. Whitman* (Silver Spring, Md.).

Hostinsky, L. Aileen. Loewy chains and uniform splitting of lattices. Proc. Amer. Math. Soc. **5**, 315–319 (1954).

Proof of a result previously announced by the author [Abstract of a thesis, Univ. of Illinois, 1949; these Rev. **11**, 154]. *P. M. Whitman* (Silver Spring, Md.).

Szász, G. Generalized complemented and quasicomplemented lattices. Publ. Math. Debrecen **3** (1953), 9–16 (1954).

Given elements a, u, v in a lattice L, $x\in L$ is called a (u, v)-complement of a in L if $a\cap x\leqq u$ and $a\cup x\geqq v$; such an x is denoted $a_u{}^v$. Then L is called generalized-complemented if every $a\in L$ has a (u, v)-complement for arbitrary $u, v\in L$. This is equivalent to the ordinary definition if L has O and I. Any relatively complemented lattice is generalized-complemented; the converse holds in modular lattices. In a generalized-complemented modular lattice, if $u\leqq a\leqq b$, v is any element, and x is a relative complement of a in $[u, b]$ then x is a (u, v)-complement of $a\cup b_u{}^v$. Then L is called quasi-complemented if every element other than O, I has a (u, v)-complement for arbitrary $u\neq O$, $v=I$. Examples are given. *P. M. Whitman.*

Maeda, Fumitomo. Matroid lattices of infinite length. J. Sci. Hiroshima Univ. Ser. A. **15**, 177–182 (1952).

A lattice of finite length is called a matroid lattice by G. Birkhoff if it is semi-modular and relatively complemented, or equivalently if it is semi-modular and relatively atomic, that is, every element is a join of atoms. Various conditions which are equivalent for lattices of finite length no longer remain so when this condition is relaxed; hence care is required in framing properly the definition of a matroid lattice of infinite length. The definition given by the author is: a matroid lattice is one which is semi-modular, relatively atomic, and upper continuous. Here semi-modular means as usual: if x and y cover z, and $x\neq y$, then $x\cup y$ covers x and y. A lattice is called upper continuous if it is complete, and if $x_\alpha\uparrow x$ implies $x_\alpha\cap y\uparrow x\cap y$. If a lattice is relatively atomic and upper continuous, the author gives five other conditions which are shown to be each equivalent to semi-modularity. All these conditions involve the covering relation. Possible definitions of semi-modularity not

involving the covering relation and applicable to non-atomic lattices are known, but are not studied here.

Since these matroid lattices of possibly infinite length form the most general setting for the notion of linear dependence, and since they are associated with linear geometries of a useful general sort, the proper definition of this concept is important. It is shown that the "exchange lattices" of S. MacLane are matroid lattices. Two of the "exchange axioms" studied by MacLane are among the formulations of semi-modularity considered. It is shown that a matroid lattice is relatively complemented.

O. Frink (State College, Pa.).

Sasaki, Usa, and Fujiwara, Shigeru. The decomposition of matroid lattices. J. Sci. Hiroshima Univ. Ser. A. **15**, 183–188 (1952).

F. Maeda has defined a matroid lattice (possibly of infinite length) to be a lattice which is relatively atomic, upper continuous, and semi-modular. The same author, after defining perspectivity and projectivity of atoms in a general lattice, showed that every relatively atomic upper continuous lattice is a direct union of sublattices such that two atoms are in the same sublattice if and only if they are projective.

In the present paper the authors show that every matroid lattice is a direct union of irreducible matroid sublattices, in such a way that two atoms are in the same irreducible sublattice if and only if they are perspective. If p and q are atoms, then p is said to be perspective to q if there exists a lattice element x such that $q \leqq p \cup x$, and $q \cap x = 0$. It is shown that in a matroid lattice this relation of perspectivity is symmetric and transitive. It is also shown that a matroid lattice is irreducible if and only if any two atoms are perspective.

O. Frink (State College, Pa.).

Sasaki, Usa. Lattice theoretic characterization of an affine geometry of arbitrary dimensions. J. Sci. Hiroshima Univ. Ser. A. **16**, 223–238 (1952).

The author shows that a lattice is isomorphic to the lattice of all linear subspaces of an affine space of arbitrary dimension, finite or infinite, if and only if it is relatively atomic, upper continuous, semi-modular, and satisfies the lattice theory equivalent of the parallel postulate. This latter condition is: if p, q, and r are independent atoms of the lattice L, then there is a unique element l of L such that $p < l < p \cup q \cup r$, and $(q \cup r) \cap l = 0$.

The author assumes a set of axioms for affine spaces similar to those of Hilbert, omitting restrictions on dimensionality. Essentially they are: two distinct points determine a line, three non-collinear points determine a plane, two planes in three-space cannot have just one point in common, and the parallel postulate. He uses the definition of semi-modularity due to Wilcox, namely that the relation $(b, c)M$ of modularity between pairs of elements is symmetric. Other definitions of semi-modularity which involve covering relations could just as well have been used, since under the conditions of his theorem they are equivalent. The proof of both the necessity and of the sufficiency of his conditions involves a great amount of detail.

O. Frink.

Sasaki, Usa. Semi-modularity in relatively atomic, upper continuous lattices. J. Sci. Hiroshima Univ. Ser. A. **16**, 409–416 (1953).

F. Maeda has defined a matroid lattice to be a relatively atomic, upper continuous, semi-modular lattice, where semi-modularity means the condition of Birkhoff: if a and b cover c, and $a \neq b$, then $a \cup b$ covers a and b. The author [see the preceding review] has characterized lattices of subspaces of affine spaces as matroid lattices obeying a lattice form of the parallel postulate. In doing so he used as the definition of semi-modularity the Wilcox formulation: the relation $(b, c)M$ of modularity between pairs of elements is symmetric, where $(b, c)M$ means: $a \leqq c$ implies $(a \cup b) \cap c = a \cup (b \cap c)$ for all b.

In this paper the author shows that in a relatively atomic upper continuous lattice the condition of Birkhoff is equivalent to that of Wilcox. Since Maeda has shown that Birkhoff's condition is equivalent to five other common formulations of modularity involving the covering relation under the same assumptions, we are now assured that in defining matroid lattices, most of the definitions that have been proposed for semi-modularity are equivalent. The author also shows that three sets of pairs of conditions which together are weaker than modularity and stronger than semi-modularity, and which are satisfied in the lattice of linear subspaces of an affine space but not in every matroid lattice, are equivalent.

O. Frink (State College, Pa.).

Maeda, Fumitomo. Direct and subdirect factorizations of lattices. J. Sci. Hiroshima Univ. Ser. A. **15**, 97–102 (1951).

Without assuming the existence of O and I elements, the author proves that any two direct factorizations of a lattice have a common refinement. To get related results for subdirect factorizations, the author introduces the notion of a canonical subdirect factorization. A subdirect factorization is determined by a set of congruence relations θ_a whose meet is the O element of the lattice T of all congruence relations. T is upper continuous, distributive, and pseudo-complemented. If $\theta_a \cup \theta_b = I$ for $a \neq b$, the factorization is called canonical. It is proved that any two canonical subdirect factorizations of a lattice have a common refinement which is also canonical. It is also shown that if a lattice L has a canonical subdirect factorization into subdirectly irreducible factors, then the lattice of all closed congruence relations on L is a complete atomic Boolean algebra. Here a congruence relation on a lattice is called closed if it is the pseudo-complement of its pseudo-complement in the lattice T of all congruence relations. This Boolean algebra is in general not a sublattice of T.

O. Frink (State College, Pa.).

Tanaka, Toshio. Canonical subdirect factorizations of lattices. J. Sci. Hiroshima Univ. Ser. A. **16**, 239–246 (1952).

F. Maeda [see the preceding review] has defined the notion of a canonical subdirect factorization of a lattice. In the present paper the author proves that a lattice L has a canonical subdirect factorization into subdirectly irreducible factors if and only if the lattice T of all congruence relations on L is atomic. He makes use of the result proved by G. Birkhoff and the reviewer [Trans. Amer. Math. Soc. **64**, 299–316 (1948); these Rev. **10**, 279] that in the lattice T, every element is the meet of completely meet-irreducible elements, and shows that every meet-irreducible element of T is either maximal or dense. As a corollary he shows that a lattice which is a dense chain has no canonical subdirect factorization into subdirectly irreducible factors.

The author also finds a solution to Birkhoff's problem 72: Find necessary and sufficient conditions on a lattice L, that its congruence relations should form a Boolean algebra. He proves that the lattice of congruence relations on L is a Boolean algebra if and only if L has a subdirect factorization

into subdirectly irreducible factors such that any two elements of L differ at most in a finite number of components. This result can be extended to the lattice of all congruence relations on certain types of algebraic systems which are not lattices. *O. Frink* (State College, Pa.).

Maeda, Fumitomo. A lattice formulation for algebraic and transcendental extensions in abstract algebras. J. Sci. Hiroshima Univ. Ser. A. 16, 383–397 (1953).

Noting an analogy between the two cases of algebraic and transcendental extensions of subfields of a field on the one hand, and the two kinds of extension of a convex set A by adjunction of a single point p, depending on whether p is or is not in the closure of A, the author seeks a common lattice-theoretical formulation including these as special cases. He considers lattices which are upper continuous and relatively molecular. A lattice is relatively molecular if every element is the join of molecular elements. A molecular element is one which is $\uparrow$-inaccessible. These conditions were shown by G. Birkhoff and the reviewer [Trans. Amer. Math. Soc. 64, 299–316 (1948); these Rev. 10, 279] to characterize lattices of all subalgebras of an algebra with finitary operations. An element x is $\uparrow$-inaccessible if $x_a \uparrow x$ implies that $x_a = x$ for some a. The author shows that an element is molecular if and only if, whenever it is the join of a set S of elements, it is the join of a finite subset of S.

In such a lattice the author introduces a notion of dependence having five formal properties due to van der Waerden. In terms of this notion of dependence he defines a closure operator on the elements of the lattice. Using this notion of dependence or the corresponding closure operator, he is able to distinguish between the two cases of extension of a lattice element by adjunction; these can be called algebraic and transcendental. For the transcendental case the transcendence degree of the extension can be defined in lattice-theoretical terms as a certain cardinal number. It is verified that the examples originally cited of extensions of subfields and of convex sets are indeed special instances of this general theory. *O. Frink* (State College, Pa.).

Molinaro, Italico. Généralisation de l'équivalence d'Artin. C. R. Acad. Sci. Paris 238, 1284–1286 (1954).

In a residuated gerbier [Dubreil-Jacotin, Lesieur, and Croisot, Leçons sur la théorie des treillis des structures algébriques ordonnées et des treillis géométriques, Gauthier-Villars, Paris, 1953; these Rev. 15, 279] with unit e and commutative multiplication, define the equivalence $\mathfrak{A}_x$ by $a \equiv b\ (\mathfrak{A}_x)$ if $x:a = x:b$. This generalizes the definition of Artin equivalence [ibid., p. 240] by having x in place of e. Properties of this relation are investigated; many are similar to the ungeneralized case. Theorems: If A is the equivalence class of a modulo $\mathfrak{A}_x$, then $x:(x:a)$ is the greatest element of A and is the only residual of x in A. $\mathfrak{A}_{x:u}$ is independent of u if and only if for all u, $u(x:xu) \equiv e\ (\mathfrak{A}_x)$; then x is called "nomal". A gerbier contains a nomal element x if and only if the set of positive idempotents has a greatest element; and then the only nomal elements are x and its residuals. A gerbier is integrally closed if and only if e is nomal. *P. M. Whitman* (Silver Spring, Md.).

Sampei, Yoemon. On lattice completions and closure operations. Comment. Math. Univ. St. Paul. 2 (1953), 55–70 (1954).

Let $\{L; \cup, \cap, \leqq\}$ be a lattice and $\{P(L); \sum, \prod, \subseteq\}$ be the lattice of all subsets of L. A mapping $a: X \to X^a$ of $P(L)$ into itself is a quasi-closure operation in L if $X \subseteq Y$ implies $X^a \subseteq Y^a$ and $X \subseteq X^a$. If, in addition, $X^{aa} = X^a$ then a is a closure operation in L. The set $Q(L)$ of all quasi-closure operations in L is a complete lattice with operations $\vee$: $a = \vee_\alpha a_\alpha$ if $X^a = \sum_\alpha X^{a_\alpha}$ and $\wedge$: $a = \wedge_\alpha a_\alpha$ if $X^a = \prod_\alpha X^{a_\alpha}$. The set $C(L)$ of all closure operations in L is a complete lattice with operations $\cap = \wedge$ and $\cup$: $\cup_\alpha a_\alpha = \overline{\vee_\alpha a_\alpha}$, where, for $a \in Q(L)$, $\bar{a}$ is the least element of $C(L) \geqq a$. Special closure operations in L are

$$m:\quad X^m = \{y; y \leqq x \text{ for some } x \in X\};$$
$$i:\quad X^i = \{y; y \leqq x_1 \cup \cdots \cup x_n \text{ for some } x_1, \cdots, x_n \in X\};$$

s: $X^s = X^{+-}$, where $X^+ = \{y;\ y \geqq x \text{ for every } x \in X\}$ and $X^- = \{y;\ y \leqq x \text{ for every } x \in X\}$. The element $a \in C(L)$ is normal if $\{x, y\}^a = \{x \cup y\}^m$ for every $x, y \in L$. The set $N(L)$ of all normal $a \in C(L)$ is a complete sublattice of $C(L)$. The least element of $N(L)$ is i. If θ is a lattice isomorphism of L into a complete lattice $\mathfrak{L}$, $(\mathfrak{L}, \theta)$ is called a completion of L. If $(\mathfrak{L}, \theta)$, $(\mathfrak{M}, \beta) \in E(L)$, the set of all completions of L, then $\mathfrak{L} \leqq \mathfrak{M}$ if there exists an order isomorphism γ of $\mathfrak{L}$ into $\mathfrak{M}$ such that $\gamma\theta = \beta$. If $(\mathfrak{L}, \theta) \in E(L)$, then the mapping $\mathfrak{L}/L$: $X \to \theta^{-1}(\{\sup \theta(X)\}^m)$, $X \in P(L)$, is in $N(L)$. For $c \in N(L)$, let $L \times c$ be the set of all $X \in P(L)$ such that $X^c = X$. Then $(L \times c, \theta) \in E(L)$ where θ: $\theta(x) = \{x\}^m$. It is proved that the mappings f: $f(c) = L \times c$ and g: $g(\mathfrak{L}) = \mathfrak{L}/L$ establish a Galois connection between $N(L)$ and $E(L)$, and that gf is the identity. There exist minimal elements in $E(L)$; all such minimal completions of L are shown to be isomorphic to MacNeille's completion $L \times s$. If p is a property of a lattice such as modularity or distributivity and $E_p(L)$ is the set of all p-completions of L, then it is shown that $E_p(L)$ has a least element if and only if $L \times s$ has property p. *R. E. Johnson* (Northampton, Mass.).

Krishnan, Viakalathur S. Closure operations on c-structures. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 317–329 (1953).

The author considers homomorphisms between very general kinds of mathematical systems called c-structures. These include as special cases systems with arbitrary operations and relations, which may also be partially ordered sets or topological spaces, the order or topology being related to the operations and relations. He considers types, which are collections of equivalent c-structures, and defines the notion of a closure operation on a type. He then proves a very general theorem about such closure operations (Theorem 3 of the paper), which is too complicated to be stated here.

Corollaries of this main theorem are: (1) A lattice is distributive if and only if it satisfies one of the equivalent conditions: it is submersible in a product of strict chains; it is the lattice product of comparable chains; it is submersible in the limit of an inverse directed class of finite Boolean algebras; it is isomorphic with the limit of an inverse directed class of finite Boolean algebras. (2) An l-group is regular if and only if it satisfies one of the equivalent conditions: it is submersible in a product of strict totally ordered groups; it is the lattice product of coarser totally ordered groups having the same base group; it is isomorphic to an order subgroup of a product of totally ordered groups; it is submersible in the limit of an inverse directed class of f-groups; it is isomorphic with the limit of an inverse directed class of f-groups. (3) A topological space is completely regular if and only if it satisfies one of the equivalent conditions: it is submersible in a product of (compact) metric spaces; it is homeomorphically inbeddable in a

product of (compact) quasi-metric spaces; the topology is the lattice product of coarser quasi-metric topologies; it is submersible in the limit of an inverse directed class of quasi-metric spaces; it is homeomorphic with the limit of an inverse directed class of quasi-metric spaces. *O. Frink.*

Copi, Irving M., and Harary, Frank. Some properties of n-adic relations. Portugaliae Math. 12, 143–152 (1953).

An n-adic relation $Rx_1x_2\cdots x_n$ is called symmetric if it admits all permutations of its arguments, and reflexive if $Rxx\cdots x$ holds for all x in the field of R. The following generalization of dyadic transitivity is given: if, with $x_{ij}=x_{ji}$ $(i, j=1, 2, \cdots, n)$, $Rx_{i1}x_{i2}\cdots x_{in}$ for $i=1, 2, \cdots, n$, then $Rx_{11}x_{22}\cdots x_{nn}$. Theorem: For even n, symmetry and transitivity imply reflexivity. Theorem: If R is reflexive, symmetric, and transitive, then so is the dyadic relation $S(xy)\equiv Rxx\cdots xy$; and, if n is odd, the field of S is that of R. Defining converse by $Rx_nx_{n-1}\cdots x_n$, and a composition by $R_1x_1x_2\cdots x_{n-1}y$, $R_kx_kx_2x_3\cdots x_{k-1}x_k\cdots x_k$ $(1<k<n)$, $R_nyx_2x_3\cdots x_n$, for some y, leads to analogues of various rules in the algebra of dyadic relations. *R. C. Lyndon.*

Lyndon, R. C. Identities in finite algebras. Proc. Amer. Math. Soc. 5, 8–9 (1954).

A finite algebraic system $\mathfrak{A}$ is constructed with the property that the set of all laws of $\mathfrak{A}$ is not a consequence of any finite subset of these laws. *B. Jónsson.*

Mori, Shinjiro. Über die Symmetrie des Prädikates "relativ prim". II. J. Sci. Hiroshima Univ. Ser. A. 15, 79–85 (1951).

The author continues a study begun in an earlier paper with the same title [same J. 14, 102–106 (1950); these Rev. 13, 101]. An ideal q in the commutative ring R is called strongly primary if every divisor of zero in the remainder class ring R/q is nilpotent and if, whenever p is the prime ideal belonging to q, then there is an element r in R such that $(r)p\subseteq q$ and r non-$\epsilon\, q$. (Every ideal which is strongly primary in the sense of Noether is strongly primary in the above sense, but the converse is not true.) The principal result is the following. Let R be a commutative ring with the properties: 1. Every proper semi-primary ideal has a prime ideal divisor distinct from R; 2. the ascending chain condition holds for prime ideals. Then every proper ideal of R has a unique representation as the intersection of finitely or infinitely many strongly primary ideals if and only if the relation "relatively prime" is symmetric in R.

F. Kiokemeister (So. Hadley, Mass.).

Mori, Shinjiro. Struktur der Multiplikationsringe. J. Sci. Hiroshima Univ. Ser. A. 16, 1–11 (1952).

A commutative ring R is called a multiplication ring if, whenever A and B are ideals of R and $A\subset B$, then there is an ideal C such that $A=BC$. Let R be a multiplication ring in which every idempotent element is the sum of finitely many mutually orthogonal primitive idempotent elements. If $R\neq R^2$, then every non-zero ideal of R is a power of R. If $R=R^2$, then each non-zero ideal of R is the product of finitely many uniquely determined prime ideals of R. Furthermore, R is the direct sum of finitely or infinitely many rings R_i where the radical of R_i (the collection of nilpotent elements) is a maximal prime ideal of R_i. The multiplication ring R is a general Z.P.I. ring [cf. Mori, same J. 10, 117–136 (1940); these Rev. 2, 121] if (1) $R\neq R^2$ and R is directly irreducible, or (2) $R=R^2$ and the ascending chain condition holds for ideals in R. *F. Kiokemeister.*

Mori, Shinziro. Über kommutative Ringe mit der Teilerkettenbedingung für Halbprimideale. J. Sci. Hiroshima Univ. Ser. A. 16, 247–260 (1952).

Let R be a commutative ring. Every ideal of R can be represented as an intersection of finitely many strongly primary ideals if and only if for each ideal q there is an upper bound for the set of lengths of all chains

$$q\subset q:(a_1)\subset q:(a_1a_2)\subset\cdots,$$

where $a_1, a_2, \cdots$ are elements of R. Let the last ideal in the finite chain $q\subset q:(a)\subset q:(a^2)\subset\cdots$ be called a limit ideal r of q. Every ideal of R can be represented as an intersection of finitely many weakly primary ideals if and only if the following conditions hold: (1) every ascending chain of semi-prime ideals is finite; (2) every chain

$$q\subset q:(a)\subset q:(a^2)\subset\cdots$$

is finite, and for each q there is an upper bound for the set of lengths of all chains $q=r_0\subset r_1\subset r_2\subset\cdots$, where r_{i+1} is a limit ideal of r_i.

The author has previously established a similar theorem for weakly primary ideals [same J. 12, 1–10 (1942); these Rev. 9, 562]. *F. Kiokemeister* (So. Hadley, Mass.).

Tominaga, Hisao. On primary ideal decompositions in non-commutative rings. Math. J. Okayama Univ. 3, 39–46 (1953).

If A and B are ideals in a ring R, then

$$AB^{-1}=\{x\in R\,|\,xB\subseteq A\},$$

and AB^{-k} is defined inductively. The quotient $B^{-k}A$ is defined similarly. If $AB^{-k-1}=AB^{-k}=B^{-k}A=B^{-k-1}A$ for some positive integer k, then AB^{-k} is called the limit ideal of A by B. If $A\subset AB^{-1}$, then B is said to be non-prime to A. The radical of a strongly primary ideal Q is a prime ideal P, and Q is said to belong to P. The prime ideal P is said to be associated with an ideal A if there exists a strongly primary ideal Q belonging to P such that $Q\neq AC^{-1}$ for some ideal C not contained in A.

Every ideal in R can be represented as the intersection of a finite number of strongly primary ideals if and only if the following conditions are satisfied. (1) The radical of any ideal is nilpotent modulo that ideal. (2) For any ideals A and B, there exists the limit ideal of A by B, and for each A there exists a finite number of ideals which, starting from A, are obtained by repeating the procedure of constructing limit ideals successively. (3) Each minimal prime divisor of an ideal A is non-prime to A. (4) If P is a prime ideal associated with an ideal A, there exists a strongly primary ideal Q belonging to P such that $A\subseteq Q$ and for any ideal B, where $B\subseteq Q$ and B non-$\subseteq A$, AB^{-1} is not a primary ideal belonging to P.

The results of this paper and the methods of proof employed are related to the work of Mori reviewed above and that of Murdoch [Canadian J. Math. 4, 43–57 (1952); these Rev. 13, 618]. *F. Kiokemeister* (So. Hadley, Mass.).

Herstein, I. N. An elementary proof of a theorem of Jacobson. Duke Math. J. 21, 45–48 (1954).

The theorem of the title is one which states that if for every element x in a ring R there is an integer $n(x)>1$ such that $x^{n(x)}=x$, then R is commutative. This result and a generalization by the author were obtained via a reduction to the case R a division ring, made possible by structure theory. Wedderburn's theorem on finite division rings and the inner conjugacy of certain isomorphic subfields of a division ring then were used to complete the proof. The

elementary character of the present proof lies in (1) the use of idempotents to obtain a simple reduction to the division-ring case, and (2) the use of factorization properties of elements of $R[x]$ with coefficients in the center Z of R to obtain (through the minimum polynomials over Z of elements of R) the result for division rings.
W. G. Lister (Providence, R. I.).

Levitzki, Jakob. Contributions to the theory of nilrings. Riveon Lematematika 7, 50–70 (1954). (Hebrew. English summary)

Let S be a ring. A sequence $\{a_i\}$ in S is said to be right vanishing (v_r) if there exists an n such that $a_1a_2\cdots a_n=0$. S is said to be v_r if every sequence is. A nilpotent ring is v_r, a v_r ring is nil, but in neither case conversely. The sum $N(S)$ of all v_r right ideals is a two-sided ideal which contains also all v_r left ideals. Let $N_1=N(S)$; let N_2 be the ideal containing N_1 such that $N_2-N_1=N(S-N_1)$, and so on transfinitely. For some ordinal τ, $N_\tau=N_{\tau+1}$, and it is shown that N_τ is the lower radical in the sense of Baer [Amer. J. Math. 65, 537–568 (1943); these Rev. 5, 88]. Similar results for left vanishing rings, and for rings which are both v_r and v_l (these latter are called v-rings).

Let $M(S)$ denote the (right) annihilator of S, the set of those a such that $Sa=(0)$. As with $N(S)$ above, a transfinite sequence can be formed, culminative in an "ultimate" right annihilator; the latter is also the intersection of all ideals A such that $M(S-A)=0$, and it is a v_l ring. Transfinite (left) powers are defined by intersection and by $S^{\tau+1}=SS^\tau$. If S is v_r, then it is (left) transfinitely nilpotent, i.e., $S^\tau=(0)$ for some τ. Moreover there exist v-rings with prescribed index of transfinite nilpotency. Theorems of similar type are proved about the "ultimate" members of the similarly defined transfinite sequences beginning with the socle and with the semi-annihilator (set of all a such that $SaS=(0)$). Also discussed are rings in which the v_r property is required not of all sequences but of some prescribed subset.

There seems to be some inconsistency in the use of "left" and "right" (possibly a consequence of the language of the article?). Thus in the text the mapping $x\to xa$ is called a left multiplication and is denoted by a_r, whereas in the summary it is denoted by a_l. *I. S. Cohen* (Cambridge, Mass.).

Ikeda, Masatosi, and Nakayama, Tadasi. On some characteristic properties of quasi-Frobenius and regular rings. Proc. Amer. Math. Soc. 5, 15–19 (1954).

In this paper properties of annihilator closures are extended from algebras to general rings. Thus the property $r(l(a_r))=a_r$ for right principal ideal a_r in an associative ring A is shown equivalent to the property that every A-left homomorphism of a principal left ideal Ab is given by multiplication on the right by an appropriate element c. A condition of this type is shown to characterize regular rings. *Marshall Hall Jr.* (Columbus, Ohio).

Krull, Wolfgang. Zur Theorie der kommutativen Integritätsbereiche. J. Reine Angew. Math. 192, 230–252 (1953).

Le §1 contient diverses propriétés des idéaux inversibles, et, en particulier, s'occupe du comportement des inverses d'idéaux par passage à des anneaux de fractions. Le §2 donne des conditions pour que les anneaux de fractions des idéaux maximaux d'un anneau normal [c.à.d. un "endlich diskret Hauptordnung" au sens de l'auteur, Idealtheorie, Springer, Berlin, 1935, §37] soient factoriels (c.à.d. admettent une unique décomposition de leurs éléments en facteurs irréductibles). Dans le §3, le plus important du mémoire, il est montré que si un anneau local régulier A est tel que son complété $\hat{A}$ est factoriel, alors A est factoriel; ceci avait été démontré par Zariski dans le cas d'un anneau local géométrique; la démonstration de Krull utilise de façon primordiale l'anneau gradué associé à A et $\hat{A}$; l'auteur montre aussi que tout anneau local A régulier et complet de dimension 2 est factoriel, et utilise pour celà les propriétés des idéaux homogènes de l'anneau gradué $K[X, Y]$ associé à A; au vu de ces résultats et de ceux d'I. S. Cohen [montrant qu'un anneau local régulier et complet est factoriel s'il est non ramifié; Trans. Amer. Math. Soc. 59, 54–106 (1946); ces Rev. 7, 509] on peut conjecturer que tout anneau local régulier est factoriel (le seul cas restant ouvert étant celui d'un anneau ramifié de dimension ≥ 3). Le §4 donne des cas où, $\mathfrak{p}$ désignant un idéal premier d'un anneau local régulier A, l'idéal $\hat{A}\mathfrak{p}$ n'a pas de composantes immergées. Le §5 étudie les extensions finies non ramifiées d'un anneau local régulier A, principalement en relations avec les anneaux de fractions $A_\mathfrak{p}$. Le §6 discute l'anneau local du sommet d'un cone quadratique. Les §§7 et 8 étudient une opération Σ sur les idéaux d'un anneau d'intégrité A: on suppose que A est l'intersection d'une famille (A_ν) d'anneaux de fractions de A, et, pour tout idéal $\mathfrak{a}$ de A, on pose $\mathfrak{a}^\Sigma=\bigcap_\nu \mathfrak{a}A_\nu$; c'est une opération ' au sens de l'auteur [loc. cit., §43]; étude des relations entre opérations Σ et opérations Λ définies à partir d'une famille multiplicativement stable d'idéaux de A. *P. Samuel* (Clermont-Ferrand).

Baker, G. A., Jr. Einstein numbers. Amer. Math. Monthly 61, 39–41 (1954).

The real field is shown to be isomorphic with the field consisting of all real numbers a such that $|a|<c$ with operations defined as follows:

$$a\oplus b=\frac{a+b}{1+ab/c^2},\quad a\odot b=\operatorname{cotanh}\left[\left(\tanh^{-1}\frac{a}{c}\right)\left(\tanh^{-1}\frac{b}{c}\right)\right].$$

The number c has the same arithmetic properties with respect to this field as ∞ has with respect to the real field.
F. Kiokemeister (So. Hadley, Mass.).

***Weyl, Hermann. A simple example for the legitimate passage from complex numbers to numbers of an arbitrary field.** Scientific papers presented to Max Born, pp. 75–79. Hafner Publishing Co. Inc., New York, N. Y., 1953. $2.50. on order

L'auteur remarque que, si un polynome $f(X_1, \cdots, X_n)$ à coéfficients entiers rationnels est tel que $f(c_1, \cdots, c_n)=0$ pour tous nombres complexes c_i, alors $f(c_1', \cdots, c_n')=0$ pour tous éléments c_i' d'un corps commutatif quelconque. Donc si un algorithme permettant d'exprimer les résultats à partir des données est valable dans le cas où celles ci sont des nombres complexes (par exemple la méthode de Lagrange pour l'inversion d'une série entière d'ordre 1), cet algorithme est valable dans un corps quelconque. Cette remarque donne une réponse (extrèmement partielle, comme l'admet l'auteur lui même) au problème de "réduction modulo p" posé par A. Weil. *P. Samuel* (Clermont-Ferrand).

Hämisch, Werner. Die Hauptnorm als Determinante über endlichrangigen Algebren mit Einselement. Math. Z. 58, 171–185 (1953).

Let $\mathfrak{A}$ be an algebra with identity element e of finite rank n over a commutative ground field $\mathfrak{K}$, and let $\mathfrak{A}_m$ be the $\mathfrak{K}$-algebra of all m-rowed square matrices with elements in $\mathfrak{A}$.

Let U be the general element of $\mathfrak{A}_m$ and let

$$\pi(t, U) = t^h - t^{h-1}\pi_1(U) + \cdots + (-1)^h \pi_h(U)$$

be the minimal polynomial of U. The coefficients of $\pi(t, U)$ are polynomials in nm indeterminates with coefficients in $\mathfrak{K}$. If A is an element of $\mathfrak{A}_m$, $\pi(t, A)$ is called the principal polynomial of A, and $\pi_h(A) = N(A)$ is called the principal norm of A [cf. N. Jacobson, The theory of rings, Amer. Math. Soc. Math. Surveys, v. 1, New York, 1943; these Rev. 5, 31]. The matrix A has an inverse if and only if $N(A) \neq 0$. The principal adjoint of U is defined to be

$$\bar{U} = \{U^{h-1} - U^{h-2}\pi_1(U) + \cdots + (-1)^{h-1}E\pi_{h-1}(U)\}(-1)^{h+1},$$

where E is the identity matrix of $\mathfrak{A}_m$. It follows that $A\bar{A} = \bar{A}A = N(A)E$, and hence that $AX = C$ implies that $N(A)X = \bar{A}C$, where

$$X' = (x_1, x_2, \cdots, x_m) \quad \text{and} \quad C' = (c_1, c_2, \cdots, c_m).$$

This is a form of Cramer's rule for m equations in m unknowns. However, the analogy can be carried no further.

Elementary determinant properties for $N(A)$ are developed: If A has a zero row or zero column, or if A has equal rows or equal columns, then $N(A) = 0$. The interchange of two rows or two columns changes the sign on $N(A)$. One may multiply any column of A on the right by an element a of $\mathfrak{A}$ and add to another column without changing $N(A)$. If B arises by multiplying a column of A on the right by a, then $N(B) = N(a)N(A)$. Corresponding statements hold for multiplication of rows on the left by a.

A factorization theorem for the principal polynomial of $\mathfrak{A}_m$ is established as well as certain properties of irreducible representations. *F. Kiokemeister* (So. Hadley, Mass.).

Springer, T. A. An algebraic proof of a theorem of H. Hopf. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 33–35 (1954).

The theorem concerned is: every commutative real division algebra (not necessarily associative, but finite-dimensional) has dimension 1 or 2 over the reals. If the algebra has a unit, it is either the field of real numbers or the field of complex numbers. The proof runs as follows: Every nonzero element of the algebra is the square of exactly two or four elements of the complex scalar extension of the algebra. Furthermore, the set of such square roots may be thought of (once a basis is introduced) as the intersection of n quadratic hypersurfaces in complex projective n-space. The points of intersection are simple points, so by Bézout's theorem the number of intersection points is 2^n. Hence $n = 1$ or 2. *D. Zelinsky* (Evanston, Ill.).

Zassenhaus, Hans. Trace functions on algebras with prime characteristic. Amer. Math. Monthly **60**, 685–692 (1953).

A function f defined on an associative algebra A over a field F of prime characteristic p with values in an algebraically closed extension field Δ of F is called a trace function on A over F if it satisfies the conditions: $f(a+b) = f(a) + f(b)$, $f(\alpha a) = \alpha f(a)$, $f(ab) = f(ba)$, and $f(a^p) = \{f(a)\}^p$ where a and b are in A and α is in F. It is proved that the trace functions on A over F coincide with the trace functions belonging to the representations of A over F by matrices of finite degree with coefficients in Δ. The number of trace functions is p^k where k denotes the number of classes of equivalent absolutely irreducible representations of A over F.

This theorem is applied to the group algebra of a finite group G. An element of G is called p-regular if its order is prime to p. An explicit construction is given for the trace functions on G for characteristic p, and it is proved that the number of classes of absolutely irreducible representations of a finite group G for characteristic p equals the number of classes of conjugate p-regular elements, a theorem previously established by R. Brauer [Über die Darstellung von Gruppen in Galoisschen Feldern, Hermann, Paris, 1935]. *F. Kiokemeister* (So. Hadley, Mass.).

Cohn, P. M. On homomorphic images of special Jordan algebras. Canadian J. Math. **6**, 253–264 (1954).

The author is concerned with the status of the set S of special Jordan algebras in the variety I of all Jordan algebras over a specified field of characteristic $\neq 2$. Let T be the subvariety of I generated by S, i.e., the set of all homomorphic images of elements of S. Then T, since it contains direct sums, subalgebras and homomorphs of its elements, can be defined by identities, and S is so definable if and only if $S = T$. This paper establishes the following. (1) $S \neq T$; in fact, there is a 3-generator special Jordan algebra with a non-special homomorph. (2) Every homomorph of a 2-generator special algebra is special. *W. G. Lister.*

✶**Chevalley, Claude C. The algebraic theory of spinors.** Columbia University Press, New York, 1954. viii+131 pp. \$3.75.

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Let M be a vector space of dimension m over a field K and Q a quadratic form on M whose associated bilinear form B defined by $B(x, y) = Q(x+y) - Q(x) - Q(y)$ is nondegenerate. This book is an account of certain groups and representations of them which are associated with Q. Special definitions and some exceptions occur if char. $K = 2$ or if K has few elements, but we ignore them here. The first chapter is preparatory, dealing generally with bilinear and quadratic forms and in particular with maximal totally isotropic subspaces and the orthogonal group G of Q. The theorem on the extendability of a Q-isomorphism of subspaces of M to an element of Q is used to obtain the conjugacy relative to G of the maximal totally isotropic subspaces of M and the index r of G is defined as their common dimension. $2r \leq m$ and, if K is algebraically closed, $r = [\frac{1}{2}m]$. The exterior algebra E over M is introduced and plays a considerable role in the sequel. The extension of the elements of G to automorphisms of E provides a representation of G whose simple components are shown to be the restrictions to the homogeneous subspaces of E.

In Chapter II the main structures appear as follows: The Clifford algebra C of Q is defined as the free associative algebra T of M modulo the relations $x^2 = Q(x)\cdot 1$, $x \in M$ (i.e., the universal associative algebra of the special Jordan algebra determined by B). The graded structure of C inherited from T is displayed and, in particular, $C = C_+ \oplus C_-$ where C_+ is the subalgebra of elements with even components. The vector spaces E and C are identified in such a way that the left multiplications differ by an anti-derivation in E. If m is even, C is a central simple algebra of dimension 2^m and C_+ is either simple with a 2-dimensional center or the direct sum of 2 central simple ideals, the alternatives depending on the discriminant of B. If m is odd C_+ is central simple, C is the tensor product of its center Z with C_+ and so may have either of the structures of C_+ in the case m even. The groups to be studied now appear as subgroups of G or of the group of nonsingular elements of C. Let Γ consist of all elements of C whose associated inner automorphisms map M into itself, and let χ be the restriction representation of Γ in M. If m is even, $\chi(\Gamma) = G$, if m is odd, $\chi(\Gamma) = G^+$,

the subgroup of elements of determinant 1. If $\Gamma^+ = \Gamma \cap C_+$, then $\chi(\Gamma^+) = G^+$. Consider the anti-automorphism of T determined by the identity on M, and let α be the anti-automorphism induced by it in C. If Γ_0 is the subgroup of all elements s of Γ with $\alpha(s) = s^{-1}$ and $\Gamma_0^+ = \Gamma_0 \cap \Gamma^+$, then $G_0^+ = \chi(\Gamma_0^+)$ is the reduced orthogonal group of Q. Then $\Gamma' \subset \Gamma_0^+$, $G' \subset G_0^+$ and if index $Q > 0$ then $G_0^+ = G'$.

If m is even, the irreducible representation ρ of C is called the spin representation and its space S is the space of spinors of Q. Then ρ induces the spin representation ρ of Γ and spin representations ρ^+, ρ_0^+ of C_+ or Γ^+, Γ_0^+ respectively. Γ generates C, so that ρ is irreducible on Γ; Γ_0^+ generates C_+, and ρ_+ may be irreducible or $S = S_1 \oplus S_2$, where S_1 and S_2 are the spaces of irreducible inequivalent representations of $C_+(\Gamma_0^+)$ called the half-spin representations.

If m is odd, C_+ has a single irreducible representation ρ and this is called the spin representation. Since Γ_0^+ generates C_+, the induced spin representations on Γ^+, Γ_0^+ are also irreducible. This representation is induced by precisely 2 representations of C in case C is not simple, and these are then the spin representations of C or Γ. The chapter concludes with a discussion of this situation over the real field including the Lie group structure on Γ and the appearance of Γ_0^+ as the simply connected covering group of G^+ in case Q is definite.

In the next section a detailed geometric study is made of the way $\rho(\Gamma)$ acts in S in case Q is of maximal index and m even. The relation of central interest is a correspondence between certain 1-dimensional subspaces of S, the pure spinors, and maximal totally isotropic subspaces of M. If Z is maximal totally isotropic, the corresponding subspace is the one annihilated by $\rho(Z)$, and conversely the pure spinors are those annihilated by some $\rho(Z)$. The subsequent investigations include: (1) determination of the condition imposed on representative spinors of Z and Z' by dim $Z \cap Z'$, (2) intrinsic characterizations of the pure spinors, (3) proof of the conjugacy of maximal totally isotropic subspaces in G_0, (4) determination of a basis for the space of bilinear invariants of ρ, (5) decomposition of $\rho \otimes \rho$ into its irreducible components. If m is odd the method of study is to imbed M in an $(m+1)$-dimensional space and suitably extend Q to a form of maximal index.

If $m = 8$ some coincidences occur, the half-spin representation spaces S_p, S_i have dimension 8, and a certain fundamental bilinear invariant of ρ is symmetric and hence the bilinear form of a quadratic form γ on S, Q and γ determine a quadratic form on $A = M \oplus S = M \oplus S_p \oplus S_i$. Also, γ, Q, and ρ permit the introduction of a commutative algebra on A and an orthogonal automorphism of this algebra permuting cyclically the three subspaces M, S_p, S_i. Through this algebra an invariant definition of the Cayley algebra on M determined by Q can be given. *W. G. Lister.*

Andreian, Cabriria I. Anneaux différentiels. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 319–332 (1951). (Romanian. Russian and French summaries)

Let $\square$ be an ordinary differential ring of which the ring of rational numbers is a subring such that the rational number 1 is a unity for $\square$; let A be any differential ideal of $\square$. The author considers differential ideals P maximal with respect to the property: $A \subseteq P$ and, for every $p \in P$, $A:p \neq A$. Every such P is prime. For each such P the set $\bar{A}_P$ of all elements x such that $A:[x] \not\subseteq P$, where $[x]$ is the differential ideal generated by x, is obviously a differential ideal. The author calls every such $\bar{A}_P$ a principal differential component of A, and proves that A is the intersection of these. The paper contains various other observations, among them one to the effect that every minimal prime ideal divisor of A is differential. *E. Kolchin* (New York, N. Y.).

Amitsur, A. S. Differential polynomials and division algebras. Ann. of Math. (2) **59**, 245–278 (1954).

Let F be a field of characteristic 0 with derivation $a \to a'$ and C be its subfield of constants. Let $F[t]$ be the ring of differential polynomials over F with $at = ta + a'$. For each $f(t) = t^n a_0 + t^{n-1} a_1 + \cdots + a_n \in F[t]$ a linear homogeneous differential equation $f(z) = z^{(n)} a_0 + z^{(n-1)} a_1 + \cdots + z a_n = 0$ is associated. $y (\neq 0) \in F$ is a solution of $f(z) = 0$ if and only if the log-derivative of y is a left root of $f(t)$. The dimension $r (\leqq n)$ of the C-module of solutions of $f(z) = 0$ in F is called the nullity of $f(t)$. Let $q(t)$ be the (least) right common multiple of all left factors of $f(t)$ similar [Ore, Ann. of Math. (2) **34**, 480–508 (1933); $\simeq$ in notation] to t. Then $q(t)$ is of degree and nullity r and $q(z) = 0$ is the minimal equation in F satisfied by all solutions of $f(z) = 0$ in F. On a vector-space $\mathfrak{V}$ over F there are introduced differential transformations (d.t.) A with $(va)A = vAa + va'$ ($v \in \mathfrak{V}$, $a \in F$). Here $\mathfrak{V}$ is considered as an $F[t]$-module with $vt = vA$, denoted as $(\mathfrak{V}, A)$. By Jacobson [ibid. **38**, 484–507 (1937)] $(\mathfrak{V}, A)$ is cyclic and $\cong F[t]/f(t)[t]$ with $f(t) \in F[t]$, a characteristic polynomial (c.p.) of A. The dimension of the C-module $\mathfrak{N}(A)$ of all $v \in \mathfrak{V}$ with $vA = 0$ is called the nullity of A, and is equal to the nullity of the adjoint

$$f^*(t) = a_0(-t)^n + a_1(-t)^{n-1} + \cdots + a_0$$

of a c.p. $f(t)$ of A. On the dual $\mathfrak{V}^*$ of $\mathfrak{V}$, the dual A^* of A is defined by $(rA^*)v = -r(vA) + (rv)'$ ($r \in \mathfrak{V}^*$, $v \in \mathfrak{V}$), and has $f^*(t)$ as a c.p. With a fixed F-basis of $\mathfrak{V}$ each d.t. is defined by a matrix. Let $e(t) = e_n(t)$ be a c.p. of the d.t. corresponding to the zero matrix, where n is the dimension of $\mathfrak{V}$. Then $e^*(t) \simeq e(t)$. A polynomial of degree n is of nullity n if and only if it is similar to $e(t)$ with n as the dimension of $\mathfrak{V}$.

Let A, B be d.t. on $\mathfrak{V}$, $\mathfrak{W}$ with c.p. $f(t)$, $g(t)$. The resultant $f(t) \times g(t)$ is defined to be a c.p. of the d.t. $A \times B$ on $\mathfrak{V} \times \mathfrak{W}$. Considering $\mathfrak{V} \times \mathfrak{W}^* = \mathrm{Hom}(\mathfrak{W}, \mathfrak{V})$, it is proved that a necessary and sufficient condition for the existence of factorizations $f(t) = f_1(t) f_2(t)$, $g(t) = g_1(t) g_2(t)$ such that $f_2(t)$ is of degree > 0 and $f_2(t) \simeq g_1(t)$ is that the nullity of $f^*(t) \times g(t)$ is > 0. Indeed, the set of all classes in $F[t]/f(t)F[t]$ having representative $h(t)$ with $h(t) g(t) \in f(t) F[t]$ constitutes a C-module of dimension equal to the nullity of $f^*(t) \times g(t)$.

The operator endomorphism ring $\mathfrak{R}(A)$ of $(\mathfrak{V}, A)$ is a C-algebra of dimension equal to the nullity of $f(t) \times f^*(t)$, and is a division algebra when $f(t)$ is irreducible. $f(t)$ is called an A-polynomial if and only if there is $\bar{f}(t)$ such that the degree and nullity of $f(t) \times \bar{f}(t)$ are equal; then this is the case with $\bar{f}(t) = f^*(t)$. The set $\mathfrak{A}(C, F)$ of all A-polynomials under the relation $\simeq$ and operation $\times$ forms a commutative semigroup with cancellation, and is mapped by $f(t) \to \mathfrak{R}(f) = \mathfrak{R}(A)$ homomorphically onto the semigroup $\mathfrak{CS}(C, F)$ of central simple algebras over C split by F, where A is a d.t. with $f(t)$ as a c.p. Write $f(t) \cong g(t)$ (resp. $f(t) \sim g(t)$) when $f(t) \simeq g(t+a)$ for some $a \in F$ (resp. $f(t) \times e_n(t) \simeq g(t) \times e_m(t)$ with some n, m). Then $\mathfrak{A}(C, F)$ under $\cong$ (resp. $\sim$) is isomorphic to $\mathfrak{CS}(C, F)$ (resp. the Brauer group $\mathfrak{B}(C, F)$). If $f(t)$ is an A-polynomial of degree n and $\mathfrak{R}(f)$ is of exponent ρ, one can find an algebraic extension D of C of degree $\leqq n$, an algebraic extension K of degree $\leqq \rho$ of the composite FD and an element $a \in F$ such that the nullity of $f(t-a)$ in K is n. If L is an extension of

F containing a solution of $f(z)=0$, then L has a subfield $L'\supseteq FE$ of form $FE(c^{1/p})$, $c \in FE$, containing n independent solutions of $f(z)=0$, where E is the constant field of L. For any extension of C linearly disjoint with F, the Brauer group of central simple algebras over C split both by F and by C is shown to be isomorphic to a subgroup of the factor group of the additive group FD^+ of FD modulo $F^+ + \mathfrak{L}(FD)$, where $\mathfrak{L}(FD)$ is the group of log-derivatives in FD, the derivation of F being extended to FD so as to map D into 0. On assuming D to be algebraic and normal over C, this subgroup is actually determined, and the result is interpreted cohomologically as well as multiplicatively. Finally, the assumption that the characteristic is 0 is examined at each phase of the theory. *T. Nakayama* (Nagoya).

Moriya, Mikao. Charakterisierung der nicht-archimedischen Bewertungen durch Grössenordnungen. Math. J. Okayama Univ. **3**, 29–38 (1953).

Let $x\rightarrow|x|$ be a non-archimedean (i.e. exponential) valuation of a division ring K, where the value group need not be commutative. Defining $x<y$ to mean $|x|\leqq|y|$, and $x\ll y$ to mean $nx<y$ for every natural number n, the following properties are evident: (O_1) $x<y$ or $y<x$ or both; (O_2) $x<y$ and $y<z$ imply $x<z$; (O_3) $x<y$ implies $xz<yz$ and $zx<zy$; (O_4) for every natural number n, $n<1$; (O_5) $x\ll 1$ and $y\ll 1$ imply $x+y\ll 1$, and there exists at least one element $\ll 1$. Such a binary relation $<$ is called a ranking ("Grössenordnung") of K. Conversely, if $<$ satisfies O_1, O_2, O_3, and O_5, the set R of all $x \in K$ such that $x<n$ for some natural number n is shown to be a valuation ring of K. The ranking determined by R coincides with $<$ if O_4 also holds. These results closely resemble those of Baer for commutative fields [S.-B. Heidelberger Akad. Wiss. **1927**, no. 8, 3–13, esp. §2]. The notion of ranking is equivalent to Baer's "order of growth". Baer defines it by means of three postulates on a "null class" N, corresponding to the set of all $x\ll 1$, then constructs the ring R, the ranking $<$, and the resulting valuation of K. *A. H. Clifford.*

Ghika, Al. Les topologies définies sur un A-module par une A-semi-norme. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 563–583 (1952). (Romanian. Russian and French summaries)

Let A be a lattice-ordered commutative ring with unit such that (i) $\alpha+(\beta\cup\gamma)=(\alpha+\beta)\cup(\alpha+\gamma)$, (ii) $\alpha>0$ implies that $\alpha(\beta\cup\gamma)=(\alpha\beta)\cup(\alpha\gamma)$, (iii) $2\alpha=0$ implies $\alpha=0$, (iv) $\alpha^2=0$ implies $\alpha=0$. A number of simple properties of such rings are given, as well as a method for introducing a topology compatible with the ring structure. Let E be an A-module. An A-semi-norm on E is an A-valued function σ on E such that $\sigma(x+y)\leqq\sigma(x)+\sigma(y)$ and $\sigma(\alpha x)=|\alpha|\sigma(x)$. A complicated set of rules is given for defining a topology in E, utilizing a given semi-norm, which is compatible with the structure of E as an A-module. For the case $A=$ an arbitrary direct product of real number fields (usual algebraic operations and order), an analogue of the Hahn-Banach theorem is obtained. *E. Hewitt.*

Theory of Groups

Thierrin, Gabriel. Sur la caractérisation des équivalences régulières dans les demi-groupes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 942–947 (1953).

The main objective of this paper is to characterize right regular equivalences in a demi-group D by means of intersections of principal right equivalences. These notions were introduced by P. Dubreil [Mém. Acad. Sci. Inst. France (2) **63**, no. 3 (1941); these Rev. **8**, 15]. If $\mathcal{H}$ is a family of complexes H_i of D, and R_{H_i} denotes the principal right equivalence defined by H_i, then $a\equiv b(R_{\mathcal{H}})$ means $a\equiv b(R_{H_i})$ for every $H_i \in \mathcal{H}$. Then $R_{\mathcal{H}}$ is evidently right regular. Conditions are given on $\mathcal{H}$ in order that $R_{\mathcal{H}}$ be right simplifiable. If R is any right regular equivalence in D which is also right simplifiable, and $\mathcal{H}$ is the family of all equivalence classes of D mod R, then $R=R_{\mathcal{H}}$. An equivalence relation R in D is called right reductive if $ax\equiv bx(R)$ for all $x \in D$ implies $a\equiv b(R)$. If $D^2=D$, every intersection of principal right equivalences is right reductive, and conversely every right regular and right reductive equivalence R is such an intersection: $R=R_{\mathcal{H}}$, with $\mathcal{H}$ the family of equivalence classes mod R. A more complicated result is stated for the case $D^2\neq D$. Proofs are promised in another paper. *A. H. Clifford* (Baltimore, Md.).

Dubreil, P. Contribution à la théorie des demi-groupes. III. Bull. Soc. Math. France **81**, 289–306 (1953).

This paper augments two earlier contributions by the author to the theory of demi-groups [Mém. Acad. Sci. Inst. France (2) **63**, no. 3 (1941); Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 183–200 (1951); these Rev. **8**, 15; **14**, 12]. Let $\mathcal{P}$ be the set of all complexes (i.e. subsets, including the empty set $\emptyset$) of a demi-group D. With respect to the usual product of complexes, and to the inclusion relation, $\mathcal{P}$ is a demi-lattice-ordered demi-group, called a sheaf ("gerbier"). For $H, K \in \mathcal{P}$, the left (right) quotient $H^{\cdot}.K$ $(H.^{\cdot}K)$ of H by K is defined to be the set of all $x \in D$ satisfying $xK\subseteq H$ $(Kx\subseteq H)$. $\mathcal{P}$ becomes thereby a residuated sheaf. $\mathcal{P}$ is also complete, with minimal element $\emptyset$, and maximal element D. A number of the results of the paper hold for any complete, residuated sheaf $\mathcal{P}$. An element F of $\mathcal{P}$ is said to be closed on the right if $FD=XD$ $(X \in \mathcal{P})$ implies $F\supseteq X$; such an F is a right ideal, i.e. $FD\subseteq F$. The following are shown to be equivalent: (1) F is closed on the right; (2) $FD^{\cdot}.D=F$; (3) F is the right residue W_H of some complex H of D. This solves a problem considered in the second paper cited above (p. 186): given a right ideal F, when does there exist a complex H of D such that $F=W_H$?

Let D be a demi-group such that $D^2\neq D$. A right ideal M of D is called broad ("large") if $M\cap D^2=M'\cap D^2$, with M' a right ideal of D, implies $M\supseteq M'$. It is shown that W_H is broad if $H\cap D^2\cap W_H=\emptyset$. This same condition suffices for the following: let R_H be the right principal equivalence defined by H; then the restriction of R_H to D^2 coincides with the right principal equivalence defined by $H\cap D^2$ within D^2. A complex K is said to be unitary on the right with respect to a complex H if $H^{\cdot}.k\subseteq H$ for every $k \in K$. Let H be a strong complex of D, and S a sub-demi-group of H. Among other results, generalizing the case $H=S$ in the first paper cited above (Theorem 16), the following is shown: S is contained in an equivalence class A mod R_H, and $A\neq W_H$; $U=S^{\cdot}.S$ is a sub-demi-group of D, unitary on the right with respect to A, and $S\subseteq U\subseteq A$. *A. H. Clifford.*

Croisot, R. Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples. Ann. Sci. Ecole Norm. Sup. (3) **70**, 361–379 (1953).

A demi-group D is said to satisfy condition (m, n), where m and n are non-negative integers, if to each $a \in D$ there exists $x \in D$ such that $a^m x a^n=a$. It is shown that, for $m+n\geqq 2$, there are only four inequivalent conditions (m, n): (I) $(m, 0)$ with $m\geqq 2$, in which case D is called right in-

versive; (II) $(0, n)$ with $n \geqq 2$, D called left inversive; (III) $(1, 1)$, D called inversive (=regular); (IV) (m, n) with $m \geqq 1$, $n \geqq 1$, and $m+n \geqq 3$. IV is equivalent to the conjunction of any two of the three conditions $(2, 0)$, $(0, 2)$, $(1, 1)$. D is right inversive if and only if every right ideal I of D is semi-prime, meaning $x^2 \varepsilon I$ implies $x \varepsilon I$, and this is so if and only if D is a class sum of disjoint right simple sub-demi-groups. IV holds if and only if every right and every left ideal of D is semi-prime, and this is so if and only if D is a class sum of disjoint groups. D is a class sum of disjoint simple sub-demi-groups S_α if and only if every two-sided ideal of D is semi-prime, and in this case the S_α are the equivalence classes modulo a regular equivalence (=congruence) relation in D. [This theorem was found independently by Olaf Andersen [Thesis, Hamburg, 1952, unpublished].]

D is said to satisfy condition (m, n): (A) with uniqueness if (m, n) holds and $a^m x a^n = a^m y a^n = a$ implies $x = y$; (B) with reciprocity if (m, n) holds and $a^m x a^n = a$ implies $x^m a x^n = x$; (C) with anti-reciprocity if (m, n) holds and $a^m x a^n = a$ and $x^m a x^n = x$ imply $x = a$. D satisfies (m, n) with uniqueness for some $m \geqq 1$, $n \geqq 1$ $(m \geqq 2, n = 0)$ if and only if D is a group (right-group). D satisfies any one of the conditions $(0, 2)$, $(2, 0)$, $(1, 1)$ with reciprocity if and only if D is completely simple. D satisfies (m, n) with anti-reciprocity for some $m \geqq 1$, $n \geqq 1$, if and only if D is a class sum of disjoint groups with commuting identity elements and with certain conditions on the orders of the elements of the groups.

A. H. Clifford (Baltimore, Md.).

McLean, David. Idempotent semigroups. Amer. Math. Monthly **61**, 110–113 (1954).

An idempotent semigroup is a system S of elements, closed under an associative, binary operation, such that $a^2 = a$ for all $a \varepsilon S$. S is called anticommutative if $ab = ba$ implies $a = b$. The author shows that if S is any idempotent semigroup, there exists a homomorphism of S onto a commutative idempotent semigroup T such that the inverse image of each element of T is an anticommutative subsemigroup of S. This theorem is then used to show that a finitely generated free idempotent semigroup is of finite order, a result proved recently by J. A. Green and D. Rees [Proc. Cambridge Philos. Soc. **48**, 35–40 (1952); these Rev. **13**, 720].

A. H. Clifford (Baltimore, Md.).

Evans, Trevor. An embedding theorem for semigroups with cancellation. Amer. J. Math. **76**, 399–413 (1954).

Let S be a cancellation semigroup with generators $g_1, g_2, \cdots$ and relations $\rho_i(g_1, g_2, \cdots) = \sigma_i(g_1, g_2, \cdots)$, both the sets being enumerable. If S does not contain any subgroup different from the unit element (if any), then S is isomorphic to the subsemigroup generated by the elements $\theta_1, \theta_2, \cdots$ of a cancellation semigroup D which is defined by the two generators a, b and the relations $\rho_i(\theta_1, \theta_2, \cdots) = \sigma_i(\theta_1, \theta_2, \cdots)$, θ_i standing for $b^i a b^i a^2 b^i$, $i = 1, 2, \cdots$. This theorem is proved by the help of eight lemmas. It cannot be generalised to semigroups with non-trivial subgroups, as is shown by a counterexample which is derived from an example due to Malcev [Math. Ann. **113**, 686–691 (1937)]. The author gives reasons for his guess that every cancellation semigroup can be embedded into a cancellation semigroup with four generators. A. M. Turing [Ann. of Math. (2) **52**, 491–505 (1950); these Rev. **12**, 239] gave an example of a finitely generated and finitely related semigroup for which the word problem is unsolvable. From this result and the above theorem, the author deduces the existence of such a semigroup with two generators.

F. W. Levi (Berlin).

Kontorovič, P. On the theory of semigroups in a group. Doklady Akad. Nauk SSSR (N.S.) **93**, 229–231 (1953). (Russian)

Let $\mathfrak{S}$ be a subsemigroup of a group G, "invariant in itself" in the sense that $a, b \varepsilon \mathfrak{S}$ implies $aba^{-1} \varepsilon \mathfrak{S}$. Using the well-known methods of Krull [Idealtheorie, Springer, Berlin, 1935, esp. p. 9], the author shows that the intersection of all the minimal prime ideals of $\mathfrak{S}$ containing a given ideal $\mathfrak{A}$ of $\mathfrak{S}$ is the radical (here called the "insulator") of $\mathfrak{A}$. If $\mathfrak{A}$ is invariant, i.e. $a \varepsilon \mathfrak{A}$, $b \varepsilon \mathfrak{S}$ imply $bab^{-1} \varepsilon \mathfrak{A}$, if Π is the set of all elements $b \varepsilon \mathfrak{S}$ such that $bx \varepsilon \mathfrak{A}$ implies $x \varepsilon \mathfrak{A}$, and if $\mathfrak{p}$ is a minimal prime ideal containing $\mathfrak{A}$, then $\mathfrak{p}$ and Π are disjoint.

A. H. Clifford (Baltimore, Md.).

Trevisan, Giorgio. Costruzione di quasigruppi con relazioni di congruenza non permutabili. Rend. Sem. Mat. Univ. Padova **22**, 11–22 (1953).

The author establishes the existence of a quasigroup with non-permutable congruence relations. This solves in part a problem posed by Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, Problem 31, p. 86; these Rev. **10**, 673].

F. Kiokemeister.

***Kurosch, A. G. Gruppentheorie. Mit einem Anhang von B. H. Neumann.** Akademie-Verlag, Berlin, 1953. xii+418 pp. DM 28.00. Have

This is a translation of the first edition [OGIZ, Moscow-Leningrad, 1944; these Rev. **9**, 267]. It is explained in the preface that the second edition [1953; these Rev. **15**, 501] appeared as the work of translation was nearing completion. To avoid further delay there was no attempt to switch to the second edition. The reviewer remarks that many algebraists will wish to have both versions on their shelf. Footnotes have been added referring to recent work, the bibliography has been brought up to 1952, and an appendix by B. H. Neumann has been added. This appendix deals largely with very recent work by the Neumanns and Higman which in particular settles a number of the questions raised by Kurosh.

I. Kaplansky (Chicago, Ill.).

Chen, Kuo-Tsai. A group ring method for finitely generated groups. Trans. Amer. Math. Soc. **76**, 275–287 (1954).

A theory is developed of invariants $V(F, R)$ derivable from a presentation of a group by generators and relations. Such invariants may be groups, as $(F, F)/(F, R)$ (Hopf), or may be rings, derived from the group ring JF or from the Magnus representation by power series. Invariants are subject to appropriate naturality conditions under homomorphism of presentations. These ideas are extended to 'B-presentations', that is, triples (F, R, α) where (F, R) is a presentation and α a homomorphism of F into a fixed group B, such that $\alpha R = 1$. As an illustration, two specific groups are distinguished by an invariant defined as follows. $O = \{\sum u_i(x_i - 1)\}$ is the fundamental ideal of JF, where J is the ring of integers, and $(R-1)_F$ the two-sided ideal generated by all $r-1$ for r in R. For every B-presentation (F, R, α), where B is the group of four elements, the value of the invariant V is defined to be the image of the quotient ring $O/(O \cap (R-1)_F)$ under the induced mapping α_1 carrying the coefficients u_i from JF into $J_3 B$.

R. C. Lyndon.

Scherk, Peter, and Kemperman, J. H. B. Complexes in abelian groups. Canadian J. Math. 6, 230–237 (1954).

Let G be an abelian group (finite or infinite), written additively, and let $A, B, \cdots$ be finite sets of elements of G. Let $[A]$ denote the number of elements in A. Let $A+B$ denote the set consisting of those elements of G that are representable as $a+b$ with a in A and b in B. It is supposed that $[A]+[B]\leqq[G]$, since otherwise $A+B=G$. The authors obtain various conditions which are sufficient to ensure that $[A+B]\geqq[A]+[B]-k$, where k is a given positive integer. Their method is based on that used by Mann [same J. 4, 64–66 (1952); these Rev. 13, 720].
H. Davenport (London).

Welter, C. P. The theory of a class of games on a sequence of squares, in terms of the advancing operation in a special group. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 194–200 (1954).

Let G be a linearly ordered countable abelian group of exponent 2 where the elements are given in their order by $1<a<b<ab<c<ac<bc<abc<d<ad<\cdots$. Let F be the operation on G without 1 to G which carries an element onto its immediate predecessor. Let G^n denote the n-fold direct sum of copies of G. Define a sequence of functions $\{\phi_n\}$ on the G^n to G by

(a) $\phi_n(1, x_1, x_2, \cdots, x_{n-1})=\phi_{n-1}(Fx_1, Fx_2, \cdots, Fx_{n-1})$,
(b) $\phi_n(xx_1, xx_2, \cdots, xx_n)=x^n\phi_n(x_1, x_2, \cdots, x_n)$

and

(c) $\phi_1(1)=1$.

In G^n let there be a sequence of n-vectors $\{y_j\}$, $j=0, 1, 2, \cdots, k$, $y_j=(y_{1j}, y_{2j}, \cdots, y_{nj})$ such that (A) y_{j+1} is formed from y_j by making precisely one change in one component of y_j, replacing it by an element to the left in the group ordering, (B) $\phi_n(y_j)=1$, $0\leqq j\leqq k$, and (C) $y_k=(1, a, b, ab, c, \cdots)$, the n-vector in which the ith component is the ith element of G under the linear ordering of G. The author proves that k always exists, given y_0, and is odd. Reference is made to R. Sprague [Math. Z. 51, 82–84 (1947); these Rev. 9, 330] and to the author [Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 304–314 (1952); these Rev. 14, 132].
F. Haimo (St. Louis, Mo.).

Fuchs, L. On abelian groups in which the classes of isomorphic proper subgroups contain the same number of subgroups. Čehoslovack. Mat. Ž. 2 (77) (1952), 387–390 (1953). (Russian. English summary)

Let all the non-trivial subgroups of an abelian group G be divided into classes of isomorphic subgroups. If each such class has precisely k members ($k>1$), then G is a finite abelian group of type (p, p) or of type (p, p, p). Reference is made to T. Szele [Acta Math. Acad. Sci. Hungar. 3, 127–129 (1952); these Rev. 14, 351], where the case $k=1$ is settled.
F. Haimo (St. Louis, Mo.).

Fuchs, L. On a special kind of duality in group theory. II. Acta Math. Acad. Sci. Hungar. 4, 299–314 (1953). (Russian summary)

[For part I see Fuchs, Kertész, and Szele, same Acta 4, 169–178 (1953); these Rev. 15, 287.] The results previously obtained for countable abelian groups are extended to groups of any cardinal number.
I. Kaplansky.

Fuchs, L. On the structure of abelian p-groups. Acta Math. Acad. Sci. Hungar. 4, 267–288 (1953). (Russian summary)

The main result gives necessary and sufficient conditions for the existence of a primary abelian group with prescribed cardinal number and Ulm invariants and is identical with a theorem proved by Kulikov [Trudy Moskov. Mat. Obšč. 2, 85–167 (1953); these Rev. 15, 9]. The author rediscovered the theorem independently. The two proofs are similar in spirit but differ in details.
I. Kaplansky.

***Russo, Salvatore. Sulla determinazione dei sottogruppi transitivi e transitivi normali del gruppo G_S con la S prodotto di due cicli dello stesso ordine.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 213–219. Casa Editrice Perrella, Roma, 1953.

In an earlier paper [Matematiche, Catania 4, 61–63 (1949); these Rev. 12, 239] the author determined the multiplication table of the group G_S, the normalizer of the permutation

$$S=(u_1u_2\cdots u_r)(u_{r+1}u_{r+2}\cdots u_{2r})$$

in the symmetric group of degree $2r$. In the present paper all transitive subgroups and all transitive normal subgroups of G_S are explicitly determined in terms of simple divisibility relations involving the order of the subgroup and certain integers which appear in its multiplication table.
S. A. Jennings (Vancouver, B. C.).

Plotkin, B. I. On nil groups. Doklady Akad. Nauk SSSR (N.S.) 94, 999–1001 (1954). (Russian)

A group satisfying the Engel condition, in the terminology of Gruenberg [Proc. Cambridge Philos. Soc. 49, 377–380 (1953); these Rev. 14, 1060], is here called a nilgroup. Every torsion-free nilgroup is an R-group. A torsion-free nilgroup satisfying the minimal condition for isolated subgroups is nilpotent.
R. A. Good (College Park, Md.).

Baer, Reinhold. Direkte Faktoren endlicher Gruppen. J. Reine Angew. Math. 192, 167–179 (1953).

The author obtains a number of properties of direct factors of finite groups which characterise them among normal subgroups. They involve commutation, and the formation of iterated Frattini subgroups, and none is simple enough to reproduce here.
Graham Higman.

Frame, J. S., and Robinson, G. de B. On a theorem of Osima and Nagao. Canadian J. Math. 6, 125–127 (1954).

Each p-block of irreducible representations of a symmetric group S_n is characterized by the associated p-core. The number l_b of (ordinary) irreducible representations in a block of weight b ($=[n-$the number of nodes in the core$]/p$) is independent of the core and is given by $l_b=\sum p_{b_1}p_{b_2}\cdots p_{b_p}$ ($\sum b_i=b$, $0\leqq b_i\leqq b$), where

$$1+p_1x+p_2x^2+\cdots=\{(1-x)(1-x^2)\cdots\}^{-1}$$

[Robinson, same J. 4, 373–380 (1952); these Rev. 14, 243]. Also the number l_b' of modular irreducible representations in it is independent of the core [Robinson, loc. cit.] and is given by $l_b'=\sum p_{b_1}p_{b_2}\cdots p_{b_{p-1}}$ ($\sum b_i=b$, $0\leqq b_i\leqq b$) [Osima, ibid. 5, 336–343 (1953); Nagao, ibid. 5, 356–363 (1953); these Rev. 15, 100; 14, 1061]. The present paper gives a version of Osima's proof to this last, which makes use of the generating functions for the above numbers and which yields, in addition, the generating functions for the numbers of p-cores and those containing a nodes.
T. Nakayama.

Dieudonné, Jean. Sur les groupes unitaires quaternioniques à deux et à trois variables. Bull. Sci. Math. (2) 77, 195–213 (1953).

This paper continues the author's study of the structure of the unitary groups [Sur les groupes classiques, Hermann, Paris, 1948; Trans. Amer. Math. Soc. **72**, 367–385 (1952); Acta Math. **87**, 175–242 (1952); these Rev. **9**, 494; **14**, 134, 239]. Let K be a generalized quaternion algebra with center Z assumed to be infinite with characteristic different from 2. Let E be an n-dimensional vector space over K and f a non-degenerate form on E which is anti-hermitian with respect to conjugation in K. The main objective here is to show in this case that $U_2(K, f)$ and $U_3(K, f)$ are isomorphic to certain other groups of unitary type. (Other cases were considered previously.) The starting point is the observation that the unitary group $U_n(K, f)$ is isomorphic to a subgroup of $O_{4n}^+(K_0, f_0)$ (K_0 a field and f_0 a symmetric, non-degenerate bilinear form). The subgroup consists of all those elements of $O_{4n}^+(K_0, f_0)$ which commute with a certain semi-linear transformation. In studying $U_3(K, f)$, use is made of the fact that $O_6^+(K_0, f_0)$ is isomorphic to certain groups in less than six variables (depending on K_0 and f_0; cf. the Acta Math. reference above). Various cases arise according to the indices of f and f_0, the latter of which, however, can always be taken greater than zero. $U_2(K, f)$ is treated by first embedding in a $U_3(K, f')$.

Finally, the following partial answer to a question left open in the cited Transactions paper is given. Let f have index $\nu=1$. Then the group T_n generated by the transvections in $U_n(K, f)$ is equal to the commutator subgroup of $U_n(K, f)$ $(n\geqq 3)$. (The case $\nu\geqq 2$ was considered in the paper mentioned.) *C. E. Rickart* (New Haven, Conn.).

Dynkin, E. B. Homological characteristics of homomorphisms of compact Lie groups. Doklady Akad. Nauk SSSR (N.S.) **91**, 1007–1009 (1953). (Russian)

The author states, without proofs, several results concerning homology in compact Lie groups. Let $P_1, \cdots, P_n$ denote the primitive cycles in the unitary group $U(n)$, introduced by Pontryagin [cf. Dynkin, same Doklady (N.S.) **91**, 201–204 (1953); these Rev. **15**, 398]. Let ξ be an invariant $(2k-1)$-differential form in $U(n)$, and let $\tilde{\xi}$ be the skew form defined by ξ in the Lie algebra of $U(n)$, or rather its extension in the complexified Lie algebra. Then

$$\int_{P_k}\xi=\frac{(2\pi i)^k(2k-2)!}{((k-1)!)^2}\tilde{\xi}(E_{11}, E_{12}, E_{21}, \cdots, E_{1k}, E_{k1})$$

holds, where the E_{ij} are the usual basic matrices. A similar formula is given for the orthogonal group in $2n$ variables. Let H be a Cartan algebra of the Lie algebra of the compact Lie group G; let $\bar{H}$ be the dual vector space. It is known [cf. Dynkin, ibid. **87**, 333–336 (1952); these Rev. **14**, 620; or Chevalley, Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 21–24; these Rev. **13**, 432] that there is a map of the space $S^k(H)$ of homogeneous polynomials of degree k in H, invariant under the Weyl group, into the primitive cohomology classes of G of dimension $2k-1$. Let $x\rightarrow\tilde{x}$ denote the map of $H_{2k-1}(G)$ into $S^k(\bar{H})$, determined by duality. Let F be a representation of G in $U(n)$, and let x be a primitive $(2k-1)$-cycle. Then it is stated, that $F(x)=n(\tilde{x}(\Lambda+\gamma)-\tilde{x}(\gamma))P_k$, where Λ is the highest weight of F, and γ half the sum of the positive roots.

For the case that the space $P_{2k-1}(G)$ of primitive $(2k-1)$-cycles is one-dimensional, the polynomial $\tilde{x}$, corresponding to $x \in P_{2k-1}(G)$, is written down in an explicit fashion. For $k=2$, this reduces of course to a multiple of the invariant quadratic form. *H. Samelson* (Princeton, N. J.).

Gel'fand, I. M., and Graev, M. I. Analogue of the Plancherel formula for real semisimple Lie groups. Doklady Akad. Nauk SSSR (N.S.) **92**, 461–464 (1953). (Russian)

The basic formula for the derivation of an extension of Plancherel's formula to the real unimodular group is obtained. It is indicated that the method applies to other real simple Lie groups. The basic formula in question represents the value $f(e)$ of a differentiable function f vanishing outside a neighborhood of the unit e of the $n\times n$ unimodular group G in terms of the integrals of f over the conjugacy classes of G. As indicated by the authors [same Doklady (N.S.) **92**, 221–224 (1953); these Rev. **15**, 601], $f(e)$ can be expressed as

$$\lim_{\lambda\to-r} c\int_G f(g)\,|\operatorname{tr}((\log g)^2)|^{\lambda/2}dg,$$

where c is a constant and r is the dimension of G. It results from this by a suitable computation that $f(e)$ is the result of applying a certain (simple and explicitly given) differential operator of order $\frac{1}{2}n(n-1)$ in the $2k$ $(k=[n/2])$ variables $\tau_1, \varphi_1, \cdots, \tau_k, \varphi_k$, where the complex eigenvalues of g are $\exp(\tau_j\pm i\varphi_j)$ $(j=1, 2, \cdots, k)$ to the product of the function of those variables obtained by integration of f with respect to complementary variables (in a local parametrization of G) and a fixed function of the $\tau_1, \varphi_1, \cdots, \tau_k, \varphi_k$, followed by evaluation at the group unit.

It is indicated that the real case is significantly more complicated than the complex case because of the existence of $k+1$ different types of conjugacy classes (which relate interestingly to the $k+1$ types of irreducible unitary representations) in the real cases as contrasted with only one in the complex case. In the case of the 2×2 unimodular group the Plancherel formula had previously been obtained by Bargmann [Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 133] and also treated by Harish-Chandra [Proc. Nat. Acad. Sci. U. S. A. **38**, 337–342 (1952); these Rev. **13**, 820]. *I. E. Segal* (New York, N. Y.).

Matsushita, Shin-ichi. Analyse harmonique dans les groupes localement compacts. I, II. C. R. Acad. Sci. Paris **237**, 955–957, 1056–1057 (1953).

The author considers the problem of the Fourier transform on a general locally compact group G from the following point of view. The transform space $\hat{G}$ is taken to be the weak closure (in the sense of $(L^1)^*$) of the set of elementary positive definite functions φ on G such that $\varphi(e)=1$, the zero function being excluded. Thus $\hat{G}$ is a locally compact space. For every $f \in L^1(G)$, the Fourier transform $\hat{f}$ on $\hat{G}$ is defined by $\hat{f}(\varphi)=\varphi(f)=\int f\varphi$. If f is restricted to the continuous functions of $L^1(G)$, then $I(\hat{f})=f(e)$ is a linear functional such that $I(\hat{f})\geqq 0$ when $\hat{f}\geqq 0$. The Hahn-Banach theorem is applied to obtain a (non-unique) Radon measure μ on $\hat{G}$ such that $f(e)=\int\hat{f}d\mu$. [However, the indicated proof of this step appears inconclusive to the reviewer.]

The mapping $f\rightarrow\hat{f}$ which appears above is linear, but fails to have the convolution properties possessed by the Fourier transform in the commutative theory, and, in particular, the Plancherel theorem cannot be formulated. In order to obtain this theorem, the author replaces the complex-valued function $\hat{f}$ by one whose value at φ is the element corresponding to f in the Hilbert space H_φ which is the completion of $L^1(G)$ with respect to the pseudo scalar product

$(g, h) = \varphi(h^*g)$. Granting the measure μ as announced above, the author's Plancherel theorem is then immediate. It asserts a linear isometry between $L^2(G)$ and a subspace of the direct integral of the Hilbert spaces H_φ with respect to μ.
L. H. Loomis (Cambridge, Mass.).

Conkling, Randall, and Ellis, David. On metric groupoids and their completions. Portugaliae Math. 12, 99–103 (1953).

Terminology: groupoid [see R. H. Bruck, Trans. Amer. Math. Soc. 60, 245–354 (1946); these Rev. 8, 134]; (uniformly) metric groupoid: a metric groupoid where the groupoid operation is (uniformly) continuous in both variables; continuity of the groupoid operation in each variable separately is denoted by the prefix "pseudo". The authors prove that a uniformly metric groupoid G is densely, isometrically, and isomorphically embeddable in a complete uniformly metric groupoid C which is abelian if and only if G is abelian and which has an identity if and only if G has one. If G is uniformly pseudometric it is a topological group [see, e.g., Banach, Studia Math. 3, 101–113 (1931); Gelbaum, Kalisch, and Olmsted, Proc. Amer. Math. Soc. 2, 807–821 (1951); these Rev. 13, 206; Koshiba, Osaka Math. J. 3, 49–53 (1951); these Rev. 12, 673; Montgomery, Bull. Amer. Math. Soc. 42, 879–882 (1936)]. Thus uniformly metric groups are topological groups and are completable to uniformly metric topological groups. Normed Abelian groups G such that $|ma| = |m||a|$ for all integers m and all $a \in G$ can be embedded in a Banach space by first embedding G in a divisible normed group (which makes it into a rational normed vector group) and then completing. The proofs are direct and straightforward.
G. K. Kalisch.

NUMBER THEORY

Have

★Piazzolla Beloch, M. Lezioni di matematica complementare. (La Matematica Elementare vista dall'alto.) Redatte dal Prof. Egidio Orzalesi. Istituto di Geometria dell'Università di Ferrara, 1953. 439 pp. 2700 Lire.

These lectures on Elementary Mathematics from a Higher Standpoint are designed primarily for the students of a secondary teachers' training course in Italy, but they may well be of interest to teachers in other countries. Summary of contents: I. Arithmetic and Algebra: Natural, rational and real numbers; divisibility, primes, congruences, continued fractions, diophantine equations; cubic and biquadratic equations; algebraic and transcendental numbers. II. Geometry: Foundations; non-euclidean geometry; congruence; area and volume; transformations; elementary geometrical problems; problems of third and fourth degree; the classical problems; theory of geometrical constructions.

Not all topics are treated in detail, but most sections are accompanied by an extensive list of references. Historical and didactical remarks add to the value of the book as teaching aid.
F. A. Behrend (Melbourne).

Rosati, Luigi Antonio. Sull'equazione diofantea
$$4/n = 1/x_1 + 1/x_2 + 1/x_3.$$
Boll. Un. Mat. Ital. (3) 9, 59–63 (1954).

It is established that the Diophantine equation of the title is solvable for integers x_1, x_2, x_3 for all n in the range $106{,}129 \leqq n \leqq 141{,}648$. This had been verified for all smaller positive $n \neq 1$ by R. Obláth [Mathesis 59, 308–316 (1950); these Rev. 12, 481].
I. Niven (Eugene, Ore.).

Kelly, John B. A characteristic property of quadratic residues. Proc. Amer. Math. Soc. 5, 38–46 (1954).

Let p be a prime of the form $4k+1$, R_p the set of quadratic residues mod p, N_p the set of quadratic nonresidues mod p, r an arbitrary quadratic residue and n an arbitrary nonresidue. Then it is known that the sets $r+N_p$ and $n+R_p$ consist of k quadratic residues and k quadratic nonresidues. Now the author proves that this property is characteristic for a prime, i.e.: if $m=4k+1$ is an integer and the residues mod m are divided in two classes A and B, where (a) $1 \in A$, (b) for every choice of $a^* \in A$, the set a^*+B contains k elements of A and k elements of B, (c) for every choice of $b^* \in B$, the set b^*+A contains k elements of A and k elements of B, then m is a prime p and A and B are the classes R_p and N_p. There is a corresponding theorem for integers of the form $4k-1$.
H. Bergström (Göteborg).

Pearson, Erna H., and Vandiver, H. S. On a new problem concerning trinomial congruences involving rational integers. Proc. Nat. Acad. Sci. U. S. A. 39, 1278–1285 (1953).

The trinomial congruence $1+ax^c \equiv by^m \pmod p$ with $c|(p-1)$, $m|(p-1)$ and $abxy \not\equiv 0 \pmod p$, p an odd prime, is considered. The following problem is proposed: If we let $mc = k(p-1)$ where k is an integer and if a, b, m, k are fixed but p and c are allowed to increase subject to the relation between them, what can be said about the number of sets of solutions of the congruence? The authors have begun a study of this problem by calculating tables that give the number of solutions in many cases; these tables seem to show, among other things, that the number of congruences with no solution increases as c increases. Certain recursion formulas that are useful for the study of the problem are also given.
H. W. Brinkmann (Swarthmore, Pa.).

Kolscher, M. Die Potenzsummen der natürlichen Zahlen. Math. Naturwiss. Unterricht 6, 307–310 (1954).

An elementary (and well-known) derivation of the formulas for the power sums $s_p = \sum_{k=0}^{n-1} k^p$ based upon a recursion formula. Certain properties of Bernoulli numbers are also mentioned.
H. W. Brinkmann.

Serpente, Guido. Sul calcolo di certe somme. Rivista Mat. Univ. Parma 4, 219–226 (1953).

This paper deals with sums of the form
$$S_{m,n} = g_0a_0^n + g_1a_1^n + \cdots + g_ma_m^n$$
where the a_i form an arithmetic progression and the g_i a geometric progression. A general formula for this sum is obtained in terms of what is called "binomial multiplication" and "binomial division" of sequences [see A. Mambriani, Ann. Mat. Pura Appl. (4) 8, 103–139 (1930)]. The formal appearance and derivation of this result are similar to that of the sum of a geometric progression (the case $n=0$) where, however, ordinary multiplication and division are employed. In the special case where $g_i = 1$ the resulting sum is expressed in terms of Bernoulli numbers.
H. W. Brinkmann (Swarthmore, Pa.).

Kemperman, J. H. B., and Scherk, Peter. On sums of sets of integers. Canadian J. Math. 6, 238–252 (1954).

Let G be an additively written commutative ordered group. I denotes the set of all $g \in G$, $0 \leqq g \leqq n$. If D is any subset of G, define $D(g) = \sum_{0<d\leqq g} 1$, where d runs through

the elements of D. Let A and B be subsets of G, $A\oplus B$ denotes the intersection of $A+B$ with I; $A\ominus B$ is the set of all $d\subset I$ for which $B+d\subset A$. Using the theorems of Mann and Khintchine's inversion principle, the authors prove several theorems, some of which are new even in the case G is the set of integers. We state only one of their theorems: Let A, B, C be finite subsets of I, $A\oplus B\subset C$ and $0\subset A$, $0\subset B$, $n\not\subset C$. Then there exists an $m\subset G$ for which

$$C(n)-C(n-m)\geqq A(m)+B(m),$$
$$m=n \text{ or } 0<2m<n,\quad n-m\subset C\ominus A,\quad n-m\subset C\ominus B.$$

P. Erdős.

Johnson, S. M. On the representations of an integer as the sum of products of integers. Trans. Amer. Math. Soc. **76**, 177–189 (1954).

Let $d_n(\nu)$ denote the number of representations of ν in the form $\nu=a_1\cdots a_n$, where $a_1, \cdots, a_n$ are positive integers, and let $R(N;n,k)$ denote the sum $\sum d_n(\nu_1)\cdots d_n(\nu_k)$ extended over all sets of positive integers $\nu_1, \cdots, \nu_k$ satisfying $\nu_1+\cdots+\nu_k=N$. Making use of the circle-method of Hardy and Littlewood, the author obtains an asymptotic formula, with estimation of remainder, for $R(N;n,k)$ when $N\to\infty$ and n, k are fixed and subject to the conditions $n\geqq 2$, $k\geqq 3$. The case $n=3$, $k=3$ is discussed in detail while the general case and a still further generalization are considered more briefly. The case $n=2$, $k=2$ had been previously investigated by A. E. Ingham [J. London Math. Soc. **2**, 202–208 (1927), pp. 207–208] and T. Estermann [Proc. London Math. Soc. (2) **31**, 123–133 (1930)] and the case $n=2$, $k=3$ by Estermann [ibid. **29**, 453–478 (1929)]. *L. Mirsky.*

Teuffel, E. Eine Rekursionsformel für Primzahlen. Jber. Deutsch. Math. Verein. **57**, Abt. 1, 34–36 (1954).

The author describes an admittedly impractical recursive method for computing the nth prime p_n, from the preceding primes $p_{n-1}, \cdots, p_1$. Let k be any even integer $\geqq 2p_{n-1}$. Then the formula for p_n is

$$p_n=[1+\{1-(1-\zeta(k)\Pi_n)^{-1}\}^{1/k}].$$

Here $[x]$ denotes the greatest integer $\leqq x$, $\zeta(s)$ is Riemann's function and is to be computed by the formula

$$2\zeta(2h)=(2\pi)^{2h}|B_{2h}|/(2h)!,$$

where B_{2h} is the Bernoulli number in the notation of Lucas which can be computed recursively along with $(2h)!$. The function

$$\Pi_n=\prod_{\nu=1}^{n-1}(1-p_\nu^{-k})=(1-p_{n-1}^{-k})\Pi_{n-1}$$

can also be found recursively and is the only place in the formula which involves p_{n-1} explicitly. This feature makes the formula superior to a formula given previously by C. Isenkrahe [Math. Ann. **53**, 42–44 (1900)].

D. H. Lehmer (Berkeley, Calif.).

Apostol, T. M. Some series involving the Riemann zeta function. Proc. Amer. Math. Soc. **5**, 239–243 (1954).

Proof of formulae such as

$$\zeta(s)(1-k^{1-s})=\sum_{n=1}^{\infty}\frac{P_n(s)\zeta(n+s)}{k^{n+s}}\,\frac{B_{n+1}(k)-B_{n+1}}{n+1},$$

where k is an integer greater than 1,

$$P_n(s)=\frac{s(s+1)\cdots(s+n-1)}{n!}$$

$B_n(x)$ is the nth Bernoulli polynomial, and B_n is the nth Bernoulli number. *T. Estermann* (London).

Atkin, A. O. L., and Swinnerton-Dyer, P. Some properties of partitions. Proc. London Math. Soc. (3) **4**, 84–106 (1954).

"We denote by $p(n)$ the number of unrestricted partitions of a positive integer n. Ramanujan discovered, and later proved, three striking arithmetical properties of $p(n)$, namely:

$$p(5n+4)\equiv 0 \pmod 5,\quad p(7n+5)\equiv 0 \pmod 7,$$
$$p(11n+6)\equiv 0 \pmod{11}.$$

All existing proofs of these results appeal to the theory of generating functions, and provide no method of actually separating the partitions concerned into q equal classes ($q=5$, 7, or 11). Dyson [Eureka no. **8**, 10–15 (1944)] discovered empirically a remarkable combinatorial method of dividing the partitions of $5n+4$ and $7n+5$ into 5 and 7 equal classes respectively. Defining the rank of a partition as the largest part minus the number of parts, he divided the partitions of any number into 5 classes according to their ranks modulo 5. For numbers of the form $5n+4$, these 5 classes are all equal, while for numbers of other forms some but not all of the classes are equal; similar results hold for 7 but definitely not for 11.

"The main object of the present paper is to prove these conjectures of Dyson. The results form part of Theorems 4 and 5. It is noteworthy that we have to obtain at the same time all the results stated in these theorems—we cannot simplify the working so as merely to obtain Dyson's identities.

"Theorems 1 to 3 give some simple congruence properties of partitions which we obtained in the course of this work. In fact, each series

$$\sum_{n=0}^{\infty}p(qn+b)y^n\quad (q=5,\ 7,\text{ or }11;\ 0\leqq b<q)$$

is congruent modulo q to a simple infinite product. Theorems 1 and 2 follow immediately from Theorems 4 and 5 respectively, but we have given direct proofs also." (From the authors' introduction.)

The authors use a method (which goes back to Ramanujan and Watson) of splitting certain series according to the residue of the exponent (mod q). They then identify the components and verify the conjectures of Dyson by means of identities among series and products connected with the theta-functions. *N. J. Fine* (Princeton, N. J.).

Gould, H. W. A note on a paper of Grosswald. Amer. Math. Monthly **61**, 251–253 (1954).

The paper contains a simple, elementary proof of a formula obtained by Grosswald [same Monthly **60**, 179–181 (1953); these Rev. **14**, 642]. The formula in question is

$$\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k}\binom{2m+k}{n}2^{-k}$$
$$=(-1)^{(n-r)/2}2^{r-n}\binom{n}{m}\binom{2m}{n}\binom{n}{r}^{-1}\quad (n+r=2m).$$

L. Carlitz (Durham, N. C.).

Carlitz, L. Some formulas of Oltramare. Math. Mag. **27**, 189–194 (1954).

By specializing the parameters in known hypergeometric identities, a large number of congruences are obtained which are similar to two formulas of Oltramare [see Dickson,

History of the theory of numbers, vol. 1, Carnegie Inst. Washington, 1919, p. 277]. As a representative example the following result may be cited:

$$\sum_{r=0}^{m-1} \frac{1}{2r+1} \frac{5\cdot 9\cdots(4r+1)}{3\cdot 7\cdots(4r-1)} \equiv (m!)^4 \pmod{4m+1},$$

where the modulus is a prime. *A. L. Whiteman.*

Carlitz, L. q-Bernoulli and Eulerian numbers. Trans. Amer. Math. Soc. **76**, 332–350 (1954).

Continuing his study of q-Bernoulli numbers and polynomials [Duke Math. J. **15**, 987–1000 (1948); these Rev. **10**, 283] the author now defines polynomials $A_{ms}=A_{ms}(q)$ by means of

$$[x]^m = \sum_{s=1}^{m} A_{ms} \begin{bmatrix} x+s-1 \\ m \end{bmatrix} \quad (m \geqq 1),$$

where $[x]=(q^x-1)/(q-1)$ and

$$\begin{bmatrix} x \\ m \end{bmatrix} = \frac{(q^x-1)(q^{x-1}-1)\cdots(q^{x-m+1}-1)}{(q-1)(q^2-1)\cdots(q^m-1)}.$$

He also defines the rational function $H_m=H_m(x,q)$ by means of the symbolic formula $(qH+1)^m=xH^m$ $(m>1)$, $H_0=1$, $H_1=1/(x-q)$. For $q=1$, A_{ms} and $H_m(x)$ reduce to well known functions, and the author derives many of the properties of the Bernoulli and related numbers from properties of the q analogues. In particular, he extends in various directions his earlier theorem of the Staudt-Clausen type. Next he derives some congruences of Kummer's type for H_m, etc. For example, if $q=a$ is integral (mod p) while x is an indeterminate, then

$$H^m(H^w-1)^r \equiv 0 \pmod{p^m, p^{re}} \quad (p^{e-1}(p-1)\,|\,w),$$

where after expansion of the left member H^k is replaced by H_k. Finally he obtains simple congruences for the numbers A_{ms}. The corresponding results for the numbers η_m and β_m defined in the author's previous paper are more complicated.

A. L. Whiteman (Los Angeles, Calif.).

Carlitz, Leonard. The first factor of the class number of a cyclic field. Canadian J. Math. **6**, 23–26 (1954).

Let R denote the rational field, p a fixed prime >3, and $R(\zeta)$ the field generated by $\zeta=\exp 2\pi i/p$ over R. Further let $K\subset R(\zeta)$ denote the cyclic field of degree a over R, where $p-1=ab$, b odd and larger than 1, and let K_0 be the largest real subfield of K/R. The author proves that the following relation holds for the relative class-field number h_a of K/K_0 (the first factor of the class number of K/R)

$$h_a \equiv 2^{-\frac{1}{2}(a-2)} \prod_n B_{bnp^n+1}, \quad \bmod p^n,\ n\geqq 1,$$

where $n=1, 3, \cdots, a-1$ and the B_m $(m=2, 4, \cdots)$ are the Bernoulli numbers in the even suffix notation. This theorem is a generalization of a theorem of Vandiver.

H. Bergström (Göteborg).

Seres, Iván. Über eine Aufgabe von Schur. Publ. Math. Debrecen **3** (1953), 138–139 (1954).

Schur conjectured that if $a_1, \cdots, a_m$ are distinct integers and $n>1$, then the polynomial

$$F(u)=\prod_{i=1}^{m}(u-a_i)^{2^n}+1$$

is irreducible. The present paper contains a very simple proof of the conjectured theorem in the case $m=2$ (it is observed that the condition $a_1\neq a_2$ is not required in this case). *L. Carlitz* (Durham, N. C.).

Dénes, Peter. Über irreguläre Kreiskörper. Publ. Math. Debrecen **3** (1953), 17–23 (1954).

Let p denote an irregular prime, r a primitive root (mod p), $q=\frac{1}{2}(p-3)$, $\zeta=e^{2\pi i/p}$, $\Omega(\zeta)$ the cyclotomic field generated by ζ, $\lambda=1-\zeta$, $\mathfrak{l}=(\lambda)$. Let $B_{m_1}, \cdots, B_{m_g}$ denote the Bernoulli numbers in the set $B_1, \cdots, B_q$ that are divisible by p and let

$$B_{ip^j}\equiv 0 \pmod{p^{2j+1}} \quad (j=0, \cdots, u_i-1),$$
$$B_{ip^{u_i}}\not\equiv 0 \pmod{p^{2u_i+1}} \quad (i=m_1, \cdots, m_g).$$

The numbers $u_1, \cdots, u_g$ are called the p-character of the Bernoulli numbers and the maximum value w is called the irregularity degree of the field $\Omega(\zeta)$ or of the prime p; if p is regular define $w=0$. It is stated that for a fixed Bernoulli number B_{m_j} the number u_{m_j} is finite; the proof is reserved for another paper.

The object of the present paper is the proof of results for irregular units analogous to theorems obtained by Kummer in the regular case. The following theorems are proved. 1. There exists in $\Omega(\zeta)$ a set of independent units $\eta_1, \cdots, \eta_q$ such that

$$\eta_i\equiv 1+\lambda^{2e_i} \pmod{\mathfrak{l}^{2e_i+1}} \quad (i=1, \cdots, q),$$

where $2e_i=u_i(p-1)+2i$ and the u_i have the same meaning as above. 2. If p is of irregularity degree w and ϵ is a unit of $\Omega(\zeta)$ such that

$$D_{2ip^{u_i}} \log \epsilon(e^v)\equiv 0 \pmod{p^{u_i+s}} \quad (i=1, \cdots, q)$$

(the symbol D denotes the derivative at $v=0$), then ϵ is the p^sth power of a unit of $\Omega(\zeta)$. 3. If the unit ϵ is congruent to a rational number (mod p^{w+s}) then ϵ is the p^sth power of a unit of $\Omega(\zeta)$. *L. Carlitz* (Durham, N. C.).

Petersson, Hans. Über automorphe Orthogonalfunktionen und die Konstruktion der automorphen Formen von positiver reeller Dimension. Math. Ann. **127**, 33–81 (1954).

In several of his previous papers [see especially S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1950**, 417–494; these Rev. **12**, 806; hereafter quoted as (P)], the author has developed a method for the construction and study of modular and some automorphic forms of positive dimensions, based on their algebraic structure, instead of the Farey dissection used by Hardy, Ramanujan, Rademacher and Zuckerman [Rademacher and Zuckerman, Ann. of Math. (2) **39**, 433–462 (1938); Zuckerman, Amer. J. Math. **62**, 127–152 (1940); these Rev. **1**, 214]. The results covered, besides modular forms, also forms of dimension $r>0$, automorphic under Fuchsian groups of the first kind, having a fundamental region with a single cusp (parabolic fixed point). In the present paper, the author further develops his method and makes it applicable to the general Fuchsian groups of first kind. Considerable use is made of the concept of orthogonality, defined by the vanishing of the inner product of two forms, $(f,g)=\int\int f(\tau)\bar{g}(\tau)y^{r-2}dxdy$, the double integral being extended over a fundamental region $\mathfrak{F}$ of the (Fuchsian, of first kind) group Γ of real, normed matrices L. If K is the class $\{\Gamma, -r, v\}$, of forms of dimension $-r$, corresponding to the multipliers $v=v(L)$, $|v|=1$; if $f\in K$, $g\in K$ and satisfy also some further conditions, then the existence (as a generalized Cauchy principal value) of (f,g) is proven. If $\mathfrak{C}^+(K)$ stands for the set of entire cusp forms of K and $\mathfrak{N}(K)$ for the set of forms of K, orthogonal to all forms in $\mathfrak{C}^+(K)$, then for every $f\in K$, there exists one and only one

$F \in \mathfrak{N}(K)$, such that $f-F \in \mathfrak{C}^+(K)$. If $-r<-2$, $\mathfrak{N}(K)$ has a normal basis of forms that behave like entire cusp-forms everywhere in $\mathfrak{F}$, except at a single point, where they have a principal part consisting of a single term in the local uniformizing variable; they can be represented by Poincaré series $G_{-r}(\tau)$ and $\Psi_{-r}(\tau, z)$ that depend, besides the indicated variables, also on the $v(L)$'s, a matrix A (where $A^{-1}\infty$ is a cusp of $\mathfrak{F}$) and an integral parameter ν. Let $\hat{K}=\{\Gamma, r-2, \bar{v}\}$ be the class complementary to K and let $F(z) \in \hat{K}$. The theorem on residues shows that the principal parts, at the poles of $F(z)$ cannot be prescribed arbitrarily, but have to satisfy a "principal parts condition". This is not only a necessary, but also a sufficient condition for the existence of an automorphic form $F(z) \in \hat{K}$ with the prescribed poles and principal parts, as follows from the Riemann-Roch theorem. The author proves the sufficiency, by actually constructing $F(z)$ and finding an explicit expression for its Fourier coefficients. For that he defines the Poincaré series $H_{-r}(\tau, v, z, A, \Gamma)$, more general than (dependence on the matrix A), but similar to the corresponding series in (P). They are cusp forms of K, in the variable τ, provided the (complex) parameter z is kept fixed. The Fourier coefficients $F_{r-2}(z)$ and $Y_{r-2}(z)$ of two different expansions of H_{-r} are functions of the variable z (observe the interchange of the role of parameter and independent variable, cf. (P)); although, in general, they are not themselves automorphic forms of $\hat{K}$ their linear combinations, $\Lambda(z)$, that satisfy the "principal-parts condition" are automorphic. If the poles and principal parts of $F(z) \in \hat{K}$ are given, they satisfy the "principal-parts condition" and it is possible to determine constant coefficients, so that the corresponding linear combination $\Lambda(z)$ of $F_{r-2}(z)$'s and $Y_{r-2}(z)$'s should have precisely those poles and principal parts. A form of $\hat{K}$, however, is completely characterized by its poles and principal parts; therefore, $\Lambda(z)=F(z)$, achieving the construction of $F(z)$. Using the relations between H_{-r} and G_{-r} and the Fourier coefficients of F_{r-2} and Y_{r-2}, the Fourier coefficients of $F(z)$ are now obtained as finite linear combinations of absolutely convergent series, similar to the Rademacher series for the partition function; in some cases they also may have arithmetical significance. The Fourier coefficients of $F(z) \in \hat{K}$ can also be considered as functionals, obtained by a linear operator H_F, uniquely defined by the principal parts of $F(z) \in \hat{K}$, and acting upon the forms of K. The paper ends with a discussion of the structural properties of the F_{r-2} and the Y_{r-2}. *E. Grosswald* (Philadelphia, Pa.).

Tsuji, Masatsugu. On lattice points in an n-dimensional ellipsoid. J. Math. Soc. Japan 5, 295–306 (1953).

Let Q denote a positive definite quadratic form in $x_1, \cdots, x_n$ ($n \geqq 2$), and let $n(r)$ be the number of lattice points inside the n-dimensional ellipsoid $Q<r^2$. Then, if $V(r)$ is the volume of the ellipsoid, we write $n(r)=V(r)+\Omega(r)$. Estimates of $\Omega(r)$ are well-known [see Landau, S.-B. Preuss. Akad. Wiss. **1915**, 458–476]. For mean values of $\Omega(r)$ estimates can be obtained which are sharper than those following trivially from estimates for $\Omega(r)$ itself. The author proves that $\int_1^r \Omega(t)t^{1-n}dt=O(1)$. His proof uses n-dimensional potential functions and an analogue of methods of Nevanlinna. The reviewer remarks that by classical methods, depending on transformation of the theta series corresponding to Q, sharper results can be obtained, e.g. by Landau's method of the paper cited above. It can be shown that the above integral even converges, and estimates can be found for $\int_r^\infty \Omega(t)t^{1-n}dt$. E.g., if $n=2$, Landau's formula (34) (an explicit formula in the form of a series with Bessel coefficients) immediately gives $\int_r^\infty = O(r^{-1/2})$. *N. G. de Bruijn.*

Tornheim, Leonard. Lattice packing in the plane without crossing arcs. Proc. Amer. Math. Soc. **4**, 734–740 (1953).

Let S be a set in a Euclidean plane and let $E(S)$ be the vector difference set $S-S$. Then a (trivial) necessary and sufficient condition, that a lattice Λ should be such that no two sets of the form $S+\lambda$, $S+\lambda'$ with λ, λ' in Λ should have a common point, is that no point of Λ other than the origin should lie in $E(S)$. The author investigates the problem of finding a similar condition that there should be no pair of (Jordan) arcs, one contained in $S+\lambda$ and the other in $S+\lambda'$, which cross each other in a suitably defined sense. The ways in which arcs can cross are studied in some detail. A point is called a local boundary point of $E(S)$ if the points x in its neighbourhood cannot be represented continuously in the form $x=y-z$ with y and z in S. It is shown that if the only points of Λ other than the origin belonging to $E(S)$ are local boundary points then there will be no arcs in $S+\lambda$, $S+\lambda'$ which cross each other at a single point. [The reader will be much puzzled if he fails to realize that A' should be read for A in (iii) of the definition of §1. Note also that A and B should be interchanged in the example following this definition.] *C. A. Rogers* (London).

Cohn, Harvey. Stable lattices. II. Canadian J. Math. 6, 265–273 (1954).

In part I of this paper [same J. **5**, 261–270 (1953); these Rev. **14**, 1066] the author defined the notion of stability of a lattice with respect to a norm function, and gave criteria for stability of the norm in algebraic number fields. In the present part further results are obtained: (i) The integer-module of $R(\cos 2\pi/N)$ has stable norm if $N \neq 1, 2, 3, 4, 6, 12$; (ii) a necessary and sufficient condition for stability of the norm in complex modules; (iii) the norm in the integer-module of the cyclotomic field $R(\exp 2\pi i/N)$ is stable if and only if N is square-free. *N. G. de Bruijn.*

Cassels, J. W. S. Über

$$\varliminf_{x\to+\infty} x|\vartheta x+\alpha-y|.$$

Math. Ann. **127**, 288–304 (1954).

Improving earlier theorems, the author shows that if α is not of the form $m\vartheta+n$ (m, n integers), then the limit of the title is at most 4/11, except when ϑ and α have the forms

$$\vartheta=(A\omega+B)/(C\omega+D),$$
$$\alpha=\Delta(-3\omega-7+14E+14F\omega)/14|C\omega+D|,$$

where $A, \cdots, F$ are integers with $\Delta=AD-BC=\pm1$, and $\omega=7^{1/2}$. In this exceptional case, the aforementioned limit is $27/28\sqrt{7}$. The proof (which involves a considerable amount of computation) depends on the following lemma. Let p_n/q_n be the convergents of the regular continued fraction expansion of ϑ, where $0<\vartheta<1$, and put $\epsilon_n=q_n\vartheta-p_n$. Then to each n there corresponds a pair of integers P_n and Q_n such that if $\alpha_n=Q_n\vartheta+\alpha-P_n$, then either $0\leqq Q_n<q_n$ and $\alpha_n(\alpha_n+\epsilon_{n-1})<0$, or $q_n \leqq Q_n<q_n+q_{n-1}$ and $\alpha_n(\alpha_n-\epsilon_n)<0$. It is asserted that the same method can be used to improve Khintchine's theorem [Math. Ann. **111**, 631–637 (1935)], that $F(\vartheta)=\max_\alpha \liminf_{x\to\infty} x|\vartheta x+\alpha-y| \leqq \frac{1}{4}$ for almost all ϑ, giving $F(\vartheta)<\frac{1}{4}$ for almost all ϑ.
W. J. LeVeque (Ann Arbor, Mich.).

Černý, Karel. Remark on Diophantine approximation. Časopis Pěst. Mat. 77, 241–242 (1952). (Czech)

The author proves that if ξ is any irrational number and $a, b, s,$ are arbitrary integers ($s>0$), then there exist infinitely many pairs of integers u, v such that $u\equiv a \pmod{s}$, $v\equiv b \pmod{s}$, and

$$\left|\xi-\frac{u}{v}\right|<\frac{(1+\epsilon)s^2}{5^{\frac{1}{2}}v^2}.$$

Here ϵ is an arbitrary positive number and the result is best possible in the sense that $5^{\frac{1}{2}}$ cannot be replaced by a larger number.

This is an improvement on a result of S. Hartman [Colloquium Math. 2, 48–51 (1949); these Rev. 12, 807], where a similar result with the right-hand number replaced by $2s^2/v^2$ is proved (see also remark by the reviewer). The proof is extremely simple, following immediately from Khintchine's result that if ξ is irrational, α real and $\epsilon>0$, then there exist infinitely many integers $p, q>0$ such that

$$|q\xi-p-\alpha|<\frac{1+\epsilon}{5^{\frac{1}{2}}q}.$$

R. A. Rankin (Birmingham).

ANALYSIS

on order

★Turán, Paul. Eine neue Methode in der Analysis und deren Anwendungen. Akadémiai Kiadó, Budapest, 1953. 196 pp. 80.00 Ft.

As the author says in his introduction, the method to which this book is devoted is not entirely new; H. Bohr seems to have been the first one who applied the theory of Diophantine approximations to problems in function theory and analytical theory of numbers. During the last 10 years, however, Turán sharpened the analytical tools and widened the range of applications. Most of the material in the book has been published before (see the quotations below), but apart from an expository lecture [Časopis Pěst. Mat. Fys. 74, 123–131 (1950); these Rev. 12, 490] this is the first systematic account.

The book consists of two parts: Part I ("Über einige neuere Aufgaben der Theorie der diophantischen Approximationen") deals with the method, Part II ("Anwendungen") with its applications. There are 6 appendices, where the author treats auxiliary theorems which are used in Part II but which are not an essential part of the main method.

Part I. The author compares the Kronecker and Dirichlet approximation theorems with their analytical equivalents. Roughly equivalent to Dirichlet's theorem is the following one (essentially used by Bohr and Landau [see Titchmarsh, The theory of the Riemann zeta-function, Oxford, 1951, Ch. 8; these Rev. 13, 741] in order to prove that $\zeta(\sigma+it)$ is not $O(\log\log t)$ in the region $\sigma>1$, $t>4$): If $a_1, \cdots, a_k$ are complex, $0<\lambda_1<\cdots<\lambda_k$, $\omega>4$, then

$$\text{(i)}\quad \max_{1\leqq t\leqq\omega^k}\left|1+\sum_{j=1}^{k}a_j\exp(2\pi i\lambda_j t)\right|>\left\{1+\sum_{j=1}^{k}|a_j|\right\}\cos 2\pi/\omega.$$

The author remarks that, for most applications, it is important to reduce rigorously the range for t rather than clinging to the best possible estimate on the right hand side of (i). Further, he considers complex values of λ also, and writes $\exp(2\pi i\lambda_j)=z_j$. Put $\sum_{j=1}^{k}a_j z_j^{\,t}=f(t)$, $\max_{1\leqq j\leqq k}|z_j|=U$, $\min_{1\leqq j\leqq k}|z_j|=u$. The "first problem" consists of finding lower estimates for $f(t)u^{-t}$; the more difficult "second problem" deals with $f(t)U^{-t}$. Typical results are as follows. If $m\geqq 0$, then (ii) there is an integer ν ($m\leqq\nu\leqq m+k$) such that

$$|f(\nu)|u^{-\nu}\geqq k^k(2e(m+k))^{-k}|a_1+\cdots+a_k|;$$

and (iii) if, moreover, $U=|z_1|\geqq\cdots\geqq|z_k|$ then there is also a ν ($m\leqq\nu\leqq m+k$) such that

$$|f(\nu)|U^{-\nu}\geqq k^k(24e^2(m+2k))^{-k}\min_{1\leqq j\leqq k}|a_1+\cdots+a_j|.$$

Part II. §1 and §2. Generalization of inequalities of Littlewood [see Turán, J. London Math. Soc. 21, 268–275 (1947); these Rev. 9, 80] and S. Bernstein on trigonometric polynomials. §3. The real roots of almost periodic trigonometric polynomials with positive coefficients [Publ. Math. Debrecen 1, 38–41 (1949); these Rev. 11, 512]. §4. Gap theorems for Dirichlet series [Hungarica Acta Math. 1, 21–29 (1947); Acta Sci. Math. Szeged 14, 209–218 (1952); these Rev. 9, 276; 14, 738]. §5. Quasianalytic functions [C. R. Acad. Sci. Paris 224, 1750–1752 (1947); these Rev. 9, 16]. §6. Approximations of analytic functions in $|z|\leqq R$ by finite sums $\sum b_j h(z\tau_j)$, where $h(z)$ is an integral function [cf. Gelfond, Mat. Sbornik N.S. 4(46), 149–156 (1938)]. §7. The asymptotic behaviour of the solutions of systems of linear differential equations. §8. Approximative solutions of differential equations. §9 and §13. Remainder term in the Prime Number Theorem [Acta Math. Acad. Sci. Hungar. 1, 48–63, 155–166 (1950); these Rev. 13, 208; 14, 137]. §10. The author proves for a general class of Dirichlet series a result analogous to his main theorem on the Quasi Riemann Hypothesis, assuming, however, an analogue of the Lindelöf hypothesis. §11 and §12. The Quasi Riemann Hypothesis [Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 197–262 (1947); these Rev. 9, 80]. §14. Carlson's theorem on the roots of the zeta function [Acta Math. Acad. Sci. Hungar. 2, 39–73 (1951); these Rev. 13, 742]. §15. The author suggests a method which may lead to a simplified proof of Linnik's theorem on the smallest prime in an arithmetic progression [see Linnik, Mat. Sbornik N.S. 15(57), 139–178 (1944); these Rev. 6, 260].

The book can be read completely without knowledge of the literature cited. The presentation is clear and the printing is excellent. *N. G. de Bruijn* (Amsterdam).

★Turán, Pál. Az analízis egy új módszeréről és annak egyes alkalmazásairól. [On a new method in analysis and on some of its applications.] Akadémiai Kiadó, Budapest, 1953. 197 pp. 60.00 Ft.

Hungarian version of the book reviewed above.

Popoviciu, Tiberiu. Considérations théoriques sur l'utilisation pratique de certaines formules d'interpolation. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 3 (1951), 441–449 (1952). (Romanian. Russian and French summaries)

The author investigates what happens to Newton's divided-difference interpolation formula when the interpolation points are numbered arbitrarily instead of in increasing order. He concludes that numbering in increasing order is best, and that the most favorable formulas are those which reduce to Euler's, Stirling's or Bessel's in the case of equidistant points. *R. P. Boas, Jr.* (Evanston, Ill.).

Ahiezer, N. I., and Bernštein, S. N. Generalization of a theorem on weight functions and application to the problem of moments. Doklady Akad. Nauk SSSR (N.S.) **92**, 1109–1112 (1953). (Russian)

The authors give a complete solution of Bernstein's approximation problem, different from (and apparently in ignorance of) that already obtained by Pollard [Proc. Amer. Math. Soc. **4**, 869–875 (1953); these Rev. **15**, 407]. The problem is to characterize those positive functions $\Phi(x)$ such that every continuous $f(x)$ which is $o(\Phi(x))$ at $\pm\infty$ can be uniformly approximated by polynomials in the topology obtained by setting $\|f\| = \sup |f(x)|/\Phi(x)$. If $\Phi(x)$ is finite on a set with a finite limit point, and $\Phi(x)\geqq 1$, the authors' necessary and sufficient conditions are as follows: there exists a sequence of polynomials $R_{2n_i}(x)$ ($n_i\uparrow\infty$), positive on the real axis and satisfying $0<\{R_{2n_i}\}^{1/2}\leqq\Phi(x)$ and $\lim_{i\to\infty}\int_{-\infty}^{\infty}(1+x^2)^{-1}\log R_{2n_i}(x)dx=\infty$. As an application, the authors give a short proof of a theorem of M. Riesz on the indeterminate case of the Hamburger moment problem. *R. P. Boas, Jr.* (Evanston, Ill.).

Ahiezer, N. I. On weak weight functions. Doklady Akad. Nauk SSSR (N.S.) **93**, 949–952 (1953). (Russian)

The author solves S. Bernstein's problem on weighted approximation by entire functions of exponential type: to characterize those positive functions $\Phi(x)$ such that every continuous function $f(x)$ which is $o(\Phi(x))$ at $\pm\infty$ can be uniformly approximated by entire functions of given exponential type p in the topology obtained by setting $\|f\| = \sup |f(x)|/\Phi(x)$. If $\Phi(x)\geqq 1$ and $\Phi(x)$ is finite on a set with a finite limit point, then $\Phi(x)$ has the property in question if and only if $\sup\int_{-\infty}^{\infty}(1+x^2)^{-1}\log|G(x)|dx=\infty$, the supremum being taken over all entire functions G of exponential type p for which $0<|G(x)|\leqq\Phi(x)$ on the real axis. *R. P. Boas, Jr.* (Evanston, Ill.).

Lorch, Lee. Derivatives of infinite order. Pacific J. Math. **3**, 773–788 (1953).

Let $f(x)$ be infinitely differentiable in a real interval (a,b). If, as $n\to\infty$, $f^{(n)}(x)\to g(x)$ uniformly or dominatedly, then $g(x)$ is necessarily of the form ke^x, where k is a constant. If $f^{(n)}(x)$ approaches a limit, as $n\to\infty$, for only one value of x, it does not necessarily do so for other values of x. However, it is known (i) that if $f(x)$ is analytic in (a,b), and $f^{(n)}(x_0)$ approaches a limit for one x_0 in (a,b), then $f^{(n)}(x)$ converges uniformly in every closed subinterval of (a,b) [G. Vitali, Rend. Circ. Mat. Palermo **14**, 209–216 (1900); V. Ganapathy Iyer, J. Indian Math. Soc. (N.S.) **8**, 94–108 (1944); these Rev. **7**, 117]. Answering the questions of Ganapathy Iyer's [loc. cit.], Boas and the reviewer [Bull. Amer. Math. Soc. **54**, 523–526 (1948); Proc. Amer. Math. Soc. **2**, 422 (1951); these Rev. **10**, 21; **13**, 17] showed (ii) that if $f^{(n)}(x)\to g(x)$ for each x in (a,b), where $g(x)$ is finite, then $f(x)$ is analytic in (a,b), hence $g(x)=ke^x$; and (iii) if $f(x)$ belongs to a Denjoy-Carleman quasi-analytic class in the open interval (a,b), and $f^{(n)}(x_0)\to l$ for one x_0 in (a,b), then $f(x)$ is analytic in (a,b). They also considered the case (iv) in which the sequence $f^{(n)}(x)$ was replaced by the weighted sequence $f^{(n)}(x)/\lambda_n$, where λ_n is a suitably chosen sequence of constants, and indicated that generalizations were possible in which the relation $f^{(n)}(x)\to g(x)$ could be interpreted in a sense more general than that of ordinary limits. In this paper the author considers limits by the method of Borel's exponential means, indicated by writing 'B-lim'. On the analogy of (i) he proves that if $f(x)$ is analytic in (a,b), and if $B\text{-}\lim_{n\to\infty}f^{(n)}(x_0)=ke^{x_0}$, for a single x_0 in (a,b), then $B\text{-}\lim_{n\to\infty}f^{(n)}(x)=ke^x$, for every x in (a,b), and discusses the case when the point x_0 and the interval (a,b) are not real. He then proves (iii) with the B-limit instead of the ordinary limit, and, pursuing (ii), gives a set of necessary and sufficient conditions that $B\text{-}\lim_{n\to\infty}f^{(n)}(x)=g(x)$, for each x in (a,b), where $g(x)$ is finite. Finally, pursuing (iv), he considers the case of weighted subsequences $f^{(\alpha n)}(x)/\lambda_n$ with ordinary limits (where α is a positive integer), and ordinary subsequences $f^{(\alpha n)}(x)$ with generalized limits. *K. Chandrasekharan.*

San Juan, Ricardo. Un contre-exemple de fonctions quasi analytiques. C. R. Acad. Sci. Paris **238**, 1185–1186 (1954).

It is known that the class of analytic functions possesses the property that, if $f(x)$ belongs to an arbitrary quasi-analytic class and $g(x)$ is analytic, and $f^{(n)}(0)=g^{(n)}(0)$ for all n, then $f(x)\equiv g(x)$. The author constructs an example of a function $g(x)$ with the same property, not analytic, but belonging to a quasi-analytic class. However, if the property is possessed by all functions of a quasi-analytic class, the class necessarily consists of analytic functions. *R. P. Boas, Jr.* (Evanston, Ill.).

Theory of Sets, Theory of Functions of Real Variables

Tarski, Alfred. Theorems on the existence of successors of cardinals, and the axiom of choice. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 26–32 (1954).

Consider the following statements: (S_1) For every cardinal $\mathfrak{m}$ there is a cardinal $\mathfrak{n}$ such that (i) $\mathfrak{m}<\mathfrak{n}$, and (ii) the formula $\mathfrak{m}<\mathfrak{p}<\mathfrak{n}$ does not hold for any cardinal $\mathfrak{p}$. (S_2) For every cardinal $\mathfrak{m}$ there is a cardinal $\mathfrak{n}$ such that (i) $\mathfrak{m}<\mathfrak{n}$, and (ii) for every cardinal $\mathfrak{p}$ the formula $\mathfrak{m}<\mathfrak{p}$ implies $\mathfrak{n}\leqq\mathfrak{p}$. ($S_3$) For every cardinal $\mathfrak{m}$ there is a cardinal $\mathfrak{n}$ such that (i) $\mathfrak{m}<\mathfrak{n}$, and (ii) for every cardinal $\mathfrak{p}$ the formula $\mathfrak{p}<\mathfrak{n}$ implies $\mathfrak{p}\leqq\mathfrak{m}$. The author shows that (S_1) can be proved without using the axiom of choice, and that (S_2) is equivalent to the axiom of choice. He remarks that no definite result is known concerning the relation between (S_3) and the axiom of choice. *F. Bagemihl* (Princeton, N. J.).

Schütte, Kurt. Kennzeichnung von Ordnungszahlen durch rekursiv erklärte Funktionen. Math. Ann. **127**, 15–32 (1954).

The author shows how ordinal functions introduced by Veblen [Trans. Amer. Math. Soc. **9**, 280–292 (1908)] can be used to set up a constructive system of notation for an initial segment of the second number class, how this system is related to Ackermann's [Math. Z. **53**, 403–413 (1951); these Rev. **12**, 579], and how such an initial segment can be mapped in a one-to-one manner onto the set of non-negative integers with the aid of their factorization into primes. *F. Bagemihl* (Princeton, N. J.).

Neumer, Walter. Zur Konstruktion von Ordnungszahlen. II. Math. Z. **59**, 434–454 (1954).

The author combines arbitrary (naïve rather than constructive) ordinals with symbols of a constructive system of notation set up in the first paper [Math. Z. **58**, 391–413 (1953); these Rev. **15**, 512] of this series for an initial segment of the second number class, and develops an algorithm

for representing ordinals by means of figures involving operators that leads to schematic normal forms for the ordinals of a certain domain, the latter being closed, in a certain sense, relative to the operators of the algorithm.

F. Bagemihl (Princeton, N. J.).

Hönig, Chaim Samuel. Proof of the well-ordering of cardinal numbers. Proc. Amer. Math. Soc. **5**, 312 (1954).

Using both the choice axiom and Zorn's theorem the author gives (without use of ordinal numbers) a short proof that any set of cardinal numbers is well-ordered: it is sufficient to consider in the combinatorial product of given sets (choice axiom used) the subsystem B of all the totally distinct elements and the projections of any maximal element of B (Zorn's theorem used). *G. Kurepa.*

Banaschewski, Bernhard. Über den Satz von Zorn. Math. Nachr. **10**, 181–186 (1953).

The author proves a theorem which is slightly stronger than Zorn's Lemma, namely: if every well-ordered subset of a partially ordered or quasi-ordered set E has an upper bound in E, then E contains a maximal element. Zorn's Lemma refers to chains of E instead of to well-ordered subsets; the author notes that an ordered set is a chain if every finite subset has a least element, and is well-ordered if every subset has a least element. He uses properties of ordinal numbers and well-ordered sets in his proof, but he does not use transfinite induction nor assume the entire theory of ordinal numbers. He states explicitly each property of ordinal numbers and well-ordered sets required for the proof, and derives most of them. He points out that there is a close analogy between his method of proof and the method of arriving at the Burali-Forti paradox.

O. Frink (State College, Pa.).

Banaschewski, Bernhard. Über die Konstruktion wohlgeordneter Mengen. Math. Nachr. **10**, 239–245 (1953).

The author examines in detail various procedures used to establish the well-ordering theorem based on the axiom of choice, and notes what principles are common to them. He compares the proofs with proofs of Zorn's Lemma and examines them in the light of J. Schmidt's theory of closure systems [Math. Nachr. **7**, 165–182 (1952); these Rev. **13**, 904]. *O. Frink* (State College, Pa.).

Cuesta, N. Ordinal deductive models. Revista Mat. Hisp.-Amer. (4) **13**, 211–223 (1953). (Spanish)

The author constructs a deductive model corresponding to an arbitrary total or partial order, as follows: Let M be a totally or partially ordered set, with ordering relation $<$. The propositions of the deductive system are the relations $a<b$, where a and b are elements of M. The proposition $a<b$ is true or false according as a does or does not precede b in the given order, so that the latter is the criterion of truth for the set of propositions under consideration. The sole deductive mechanism is the transitive law: the proposition $a<c$ is deducible from the propositions $a<b$ and $b<c$. The demonstrability of a proposition is taken to mean its obtainability, by means of the deductive mechanism, from finitely many (at least two) propositions, called premises of the original proposition. Consequently, if $a<b$ is a true proposition, and if there is no $x \in M$ such that $a<x<b$, then $a<b$ has no true premises, is therefore indemonstrable by means of true propositions, and is called an axiom. If, however, the given order is dense, then every true proposition is demonstrable by means of true propositions without incurring a vicious circle, and there are no axioms. For a particular order it may be the case that we do not have rules enabling us to recognize all the elements of the set T of true propositions, but that only a proper subset T_1 is known or given to us. Then the truth of the propositions of only the set T_ω, consisting of T_1 and those propositions of T that can be reached from T_1 in a finite number of deductive steps, can be "investigated deductively" with T_1 as basis; the propositions in $T-T_\omega$ are "transcendentally true" with respect to T_1. Examples are given of deductive systems corresponding to several special orders. *F. Bagemihl.*

Lyapunov, A. A. On criteria of degeneracy of R-sets. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 563–578 (1953). (Russian)

The paper gives proofs of several propositions some of which were announced earlier [Doklady Akad. Nauk SSSR (N.S.) **58**, 1887–1890 (1947); these Rev. **9**, 339]. It is especially concerned with the question of finding in what measure the degeneracy criteria (Suslin, Luzin, etc.) for A-sets to yield B-sets have analogues for R-sets. Contrary to what happens with the Suslin criterion, the Luzin uniqueness criterion is transferable (Th. 4). Therefore, the "rigid" bases N of δp-operations are studied [according to Otchan, Mat. Sbornik N.S. **10**(52), 151–163 (1942); these Rev. **7**, 8] (N is rigid, provided it contains no two distinct comparable chains). In §2 one proves that (like A) R^α, R_c^α are operations definable by means of rigid bases (not so are lim inf, lim sup), a consequence of which is a form of uniqueness statement. Given: rigid basis N, the sequence $\{E_n\}_n$ of sets; then a point x is called an N-unique point of $\{E_n\}_n$ if there is a unique chain η of N so that $x \in \bigcap E_n$ $(n \in \eta)$. Here is the analogue of Luzin's uniqueness statement. If $[N]$ is a rigid basis of R^α, let $\mathcal{E}=\{E_{n_1\cdots n_k}\}$ be a table or matrix of BR_α-sets (resp. $BR_{\alpha\beta}$-sets); if $U=R_{[N]}(\{E_{n_1\cdots n_k}\})$ equals the set of the $[N]$-unique points relative to $\mathcal{E}$, then U is a BR_α-set (resp. $BR_{\alpha\beta}$-set) (Th. 4). The Glivenko imbedding theorem [ibid. **36**, 138–142 (1929)] is extended as follows: Under the same suppositions, there exists a set mapping $\mathcal{E}'=\{H_{n_1\cdots n_k}\}$ of BR_α-sets (resp. $BR_{\alpha\beta}$-sets) so that $H_{n_1\cdots n_k}\supseteq E_{n_1\cdots n_k}$ and that each point of $R_{[N]}(\mathcal{E}')$ is an $[N]$-unique point of $\mathcal{E}'$ (Th. 5). *G. Kurepa.*

Sierpiński, W. Sur une propriété des ensembles analytiques linéaires (solution d'un problème de E. Marczewski). Fund. Math. **40**, 171 (1953).

Soit F la famille de tous les ensembles plans dont chacun est l'intersection d'une suite dénombrable dont les termes sont réunion d'un nombre fini de rectangles aux côtes parallèles aux axes des coordonnées. Alors, tout ensemble analytique linéaire borné est la projection orthogonale d'un élément de F [cf. Souslin, C. R. Acad. Sci. Paris **164**, 88–91 (1917), Th. IV]. *G. Kurepa* (Zagreb).

Schwabhäuser, Wolfram. Zur Definition des geordneten Paares von Mengen beliebiger Stufe. Math. Nachr. **11**, 81–84 (1954).

There are several methods, such as those due to Wiener, Hausdorff, and Kuratowski, of defining the notion of an ordered pair $[a, b]$ in terms of the notion of an unordered pair (a, b). The author wishes to modify the definition due to Kuratowski, namely $[a, b]=((a, b), a)$ so that it conforms to a strict theory of types, which requires that all the elements of a set be of the same type. He succeeds in doing this by the use of a standard procedure for raising the type

of a set, and in such a way that the theorem: $[a, b]=[c, d]$ implies that $a=c$ and $b=d$, holds. *O. Frink.*

Popadić, Milan S. On ordered sets with finite chains. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **5** (1952), no. 1, 8 pp. (1954). (Serbo-Croatian. English summary)

A necessary and sufficient condition is derived for a partially ordered set to have the property that all its simply ordered subsets are finite. This condition is formally the same as a necessary and sufficient condition obtained by the author [same Annuaire **4**, no. 6 (1951); these Rev. **14**, 733] for a simply ordered set to be finite. *F. Bagemihl.*

Sierpiński, W. Un théorème concernant les fonctions continues dans les ensembles ordonnés. Ann. Soc. Polon. Math. **24** (1951), no. 2, 175–180 (1954).

Let E be an ordered set, $f(E)$ a function defined on the elements of E whose values lie in the ordered set H. The author proves that if E is denumerable then $f(E)$ is the limit of a sequence of continuous functions. (The continuity is defined in the order topology.) This generalises a previous result of the author [Fund. Math. **38**, 204–208 (1951); these Rev. **13**, 828]. *P. Erdös* (South Bend, Ind.).

★Sierpiński, Wacław. On the congruence of sets and their equivalence by finite decomposition. Lucknow University Studies, no. xx. The Lucknow University, Lucknow, 1954. 117 pp.

This is a charming little book, rambling pleasantly over the subject of its title and, incidentally, requiring almost no mathematical knowledge on the part of the reader beyond familiarity with some elementary metric properties of Euclidean space. Most of the material was presented by the author in a series of lectures at the University of Lucknow in 1949. It contains discussions of the paradoxes associated with the names of Hausdorff, von Neumann, and Banach and Tarski, and of the relation of these paradoxes to the existence of measures of various kinds. There are several other results and some unsolved problems; the flavor of the whole is perhaps best communicated by giving a sample of two of each. For this purpose, write $A\cong B$ whenever A and B are plane sets that are superposable by a translation or a rotation, and write $A\sim B$ whenever A and B can be decomposed into the same finite number of disjoint sets A_i and B_i such that $A_i\cong B_i$. Theorem: if E is linear set, then there is at most one point p in E such that $E-\{p\}\cong E$. Problem: does there exist a non-empty plane set E such that $E-\{p\}\cong E$ for all p in E? Theorem: if T is an isosceles right triangle and S is a square of the same area, then $S\sim T$. Problem: if C is a circle and S is a square of the same area, then is it true that $C\sim S$? *P. R. Halmos.*

Hadwiger, H. Zum Problem der Zerlegungsgleichheit k-dimensionaler Polyeder. Math. Ann. **127**, 170–174 (1954).

By construction of additive polyhedron-functionals in k-dimensional euclidean space the author obtains necessary conditions for the equivalence by decomposition of two polyhedra. For $k=3$, the conditions are equivalent to the classical conditions of Dehn [Math. Ann. **55**, 465–478 (1902)]. *B. Jessen* (Copenhagen).

Adams, J. F. On decompositions of the sphere. J. London Math. Soc. **29**, 96–99 (1954).

L'auteur considère le problème de la décomposition d'un sphère par rapport aux congruences obtenues par les rotations et la réflexion, et il démontre le théorème suivant: étant donné un ensemble X de congruences telles que

$$A_{i_1}+A_{i_2}+\cdots+A_{i_r}\cong A_{j_1}+A_{j_2}+\cdots+A_{j_s}$$

sur les sous-ensembles A_k $(k=1, 2, \cdots, n)$ d'un sphère S, on peut décomposer S en sous-ensembles A_k disjoints comme ils remplissent les congruences de X. C'est une modification d'un théorème de R. M. Robinson [Fund. Math. **34**, 246–260 (1947); ces Rev. **10**, 106] et comme un cas spécial, il admet une décomposition de S en les ensembles A, B, et C disjoints de manière que

$$A\cong B\cong C\cong A+B\cong B+C\cong C+A.$$

M. Kondô (Tokyo).

Volkmann, Bodo. Über Hausdorffsche Dimensionen von Mengen, die durch Zifferneigenschaften charakterisiert sind. IV. Math. Z. **59**, 425–433 (1954).

[For parts I–III see Math. Z. **58**, 284–287; **59**, 247–254, 259–270 (1953); these Rev. **14**, 1070; **15**, 513.] For g-adic representations $\rho=\sum e_i g^{-i}$, the possible digits e $(0\leqq e<g)$ are assigned non-negative weights $\lambda(e)$ and are divided into exclusive classes $G_1,\cdots,G_m$. Define $S_\mu(\rho, n)=\sum\lambda(e_i)$ $(i\leqq n,\ e_i\in G_\mu)$. The author determines the Hausdorff dimension of the set $E(\lambda; G; \zeta)$ of fractions ρ for which $S_\mu\sim\zeta_\mu n$ (all μ) as $n\to\infty$. *H. D. Ursell* (Leeds).

Marstrand, J. M. The dimension of Cartesian product sets. Proc. Cambridge Philos. Soc. **50**, 198–202 (1954).

E is a plane set, E_x its section by a line $x=\text{const}$. If $\Lambda^t E_x\geqq p$ for all $x\in X$ it is proved that $\Lambda^{s+t}E\geqq kp\Lambda^s X$, k a constant. In particular, $\Lambda^{s+t}(A\times B)\geqq k\Lambda^s A\cdot\Lambda^t B$.

H. D. Ursell (Leeds).

Novák, Josef, and Novotný, Miroslav. On the convergence in σ-algebras of point-sets. Čehoslovack. Mat. Ž. **3**(78), 291–296 (1953). (Russian summary)

The principal result is that if A is a Boolean σ-algebra of sets then a necessary and sufficient condition that A be metrizable in such a way that metric convergence coincides with set-theoretic convergence (lim sup = lim inf) is that A be isomorphic to the class of all subsets of a countable set. *P. R. Halmos* (Chicago, Ill.).

Mařík, Jan. Foundations of the theory of the integral in Euclidean spaces. Časopis Pěst. Mat. **77**, 1–51, 125–145, 267–301 (1952). (Czech)

Mařík, Jan. Abstract of the article "Foundations of the theory of integration in Euclidean spaces". Čehoslovack. Mat. Ž. **2**(77), 273–277 (1952). (Russian. English summary)

This is a careful exposition of the theory of the Perron-Stieltjes integral in several dimensions, apparently designed for readers with modest mathematical attainments, since many elementary facts are painstakingly stated, and no knowledge of advanced topics is assumed (for example, the Lebesgue integral is nowhere mentioned). Part I is of an introductory nature. In Part II, the Perron-Stieltjes integral in several dimensions is defined. An interval is a non-degenerate closed m-dimensional parallelotope. Let K be an m-dimensional interval. An extended-real-valued function G defined for all subintervals of K (an interval function on K) is said to be superadditive (subadditive, additive) if $G(I\cup J)\geqq G(I)+G(J)$ $(\leqq, =)$ for all intervals $I, J\subset K$ such that $I\cup J$ is an interval, I and J have disjoint interiors, and $G(I)+G(J)$ is not of the form $\infty-\infty$. If F is an interval function on K and G is a finite non-negative interval func-

tion on K, the upper derivative $\bar{F}(G, x, K)$ of F with respect to G at $x \in K$ is the supremum of all numbers $\lim_{n\to\infty} F(I_n)/G(I_n)$, where $x \in I_n$ and diam $(I_n)\to 0$. The lower derivative $\underline{F}(G, x, K)$ is defined as the infimum of all these numbers. If both $F(I)$ and $G(I)$ are zero for all sufficiently small intervals containing x, then neither derivative exists. A number of elementary properties of these derivatives are stated and proved. Let f be a point function defined on K and let G be a finite, non-negative, additive interval function on K. A superadditive interval function M on K is a majorant of f if $-\infty \neq \underline{M}(G, x, K) \geq f(x)$ for all $x \in K$. The upper Perron-Stieltjes integral of f, written $\bar{\int}_K f dG$, is the infimum of all $M(K)$, where M runs through all majorants of f. The lower integral $\underline{\int}_K f dG$ is the supremum of $N(K)$, where N runs through all minorants of f. If the upper and lower integrals are equal and finite, their common value is the Perron-Stieltjes integral of f with respect to G over K, written $\int_K f dG$. A large number of elementary theorems concerning these integrals are given. For example, it is shown that if the Riemann-Stieltjes integral exists, then so does the Perron-Stieltjes integral, and both are equal. Also, if f_n are Perron-Stieltjes integrable $(n=1, 2, \cdots)$ and $f_n \uparrow f$, then $\underline{\int}_K f dG = \bar{\int}_K f dG = \lim \int_K f_n dG$. Fubini's theorem is next proved, in the following form. Let K (L) be m-$(n$-$)$ dimensional intervals, and let G (H) be non-negative interval functions on K (L). Then the interval-function Γ defined on $K\times L$ by $\Gamma(I\times J) = H(I)G(J)$ is additive, superadditive, or subadditive, provided that G and H have these properties, respectively. If $f(x, y)$ is Perron-Stieltjes integrable on $K\times L$ with respect to Γ, then

$$\int_K\left[\bar{\int}_L f(x, y)dH(y)\right]dG(x) = \int_K\left[\underline{\int}_L f(x, y)dH(y)\right]dG(x) = \int_{K\times L} f(x, y)d\Gamma.$$

A variety of special one-dimensional theorems are proved. Integration by parts is justified, and indefinite integrals of various kinds are discussed. Part III treats absolutely convergent integrals (sic). It is shown, for example, that f and $|f|$ are Perron-Stieltjes integrable if and only if for every $\epsilon > 0$, there exists a continuous function ϕ such that $\bar{\int}_K |f-\phi| dG < \epsilon$. Interval functions of finite variation are examined in great detail. The theory of measure, based on Perron-Stieltjes integrals of characteristic functions of sets, is also set forth. 36 exercises, some of them decidedly nontrivial, are provided. *E. Hewitt* (Seattle, Wash.).

Ionescu Tulcea, C. T. Sur l'intégration des fonctions d'ensemble. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 75–83 (1952). (Romanian. Russian and French summaries)

The author defines an integral for functions defined on certain subsets of a given abstract set and with values in a separable abelian topological group. The integral is a natural generalization of an integral considered by the reviewer [Trans. Amer. Math. Soc. **52**, 498–521 (1942); these Rev. **4**, 162] for functions with values in a convex linear topological space. Most of the fundamental properties of the integral are obtained in the more general situation, the only important exception being the general convergence theorem. (It seems plausible that an appropriate form of the convergence theorem might also go through.) The concept of a filter is used to good advantage here and seems to simplify matters somewhat. *C. E. Rickart* (New Haven, Conn.).

Krickeberg, Klaus. Über den Gaussschen und den Stokesschen Integralsatz. II. Math. Nachr. **11**, 35–60 (1954).

Let f be a mapping of an m-dimensional compact orientated manifold M into an n-dimensional manifold N. Let the boundary of M be ∂M and let ω be a differential form of degree $m-1$ defined over N and with the derivative $\delta\omega$. Then it has been shown by de Possel [Bull. Sci. Math. (2) **62**, 262–271 (1938)] that, provided certain conditions on M, N, f are satisfied, there is an equality of integrals which generalises Stokes's formula, namely $\int_{f(M)} \delta\omega = \int_{f(\partial M)} \omega$. In the present paper the same formula is proved under weaker conditions. The larger part of the paper is devoted to the necessary preliminaries and definitions. In the previous paper of this series on Gauss's formula [Math. Nachr. **10**, 261–314 (1953); these Rev. **15**, 611] the author considered how far his conditions were necessary. In this paper no attempt is made to prove the necessity of the conditions used. *H. G. Eggleston* (Cambridge, England).

Froda, Al. Sur quelques propriétés des fonctions vectorielles d'une variable réelle. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 157–175 (1951). (Romanian. Russian and French summaries)

The author considers real three-dimensional vector functions $\mathbf{V}(t)$ of a real variable t. The first chapter contains straightforward generalizations of known theorems on functions of bounded variation, measurable functions, and differentiable functions. In the second chapter the author defines a vector as having positive, negative, or neutral orientation according as its angle with a certain fixed axis is acute, right, or obtuse. He considers the sequence $\mathbf{V}_j = \mathbf{V}(t_j)$, where t_j is a real sequence, and he says that the sequence $\mathbf{V}_j$ changes its orientation infinitely many times if there exists any axis with respect to which this is the case. He calls direct and retrograde vectors of accumulation at t_0 the limit vectors obtained by letting t_j run over monotone decreasing or increasing sequences which approach t_0. He then derives the following theorems. I. If among the direct (retrograde) vectors of accumulation of $\mathbf{V}(t)$ at t_0 there exist two which subtend a non-zero angle, then there exists a monotone decreasing (increasing) sequence t_p such that $\mathbf{V}_p$ changes its orientation infinitely many times. II. If among the direct (retrograde) derivative vectors of $\mathbf{V}(t)$ at t_0, supposed of bounded variation, there exist two which subtend a non-zero angle, then there exists a monotone decreasing (increasing) sequence t_p approaching t_0 and a vector function $\mathbf{W}(t)$ such that $\mathbf{W}(t) = d\mathbf{V}(t)/dt$ and $\mathbf{W}_p$ changes its orientation infinitely many times. *C. Truesdell.*

Livingston, Arthur E. The zeros of a certain class of indefinite integrals. Proc. Amer. Math. Soc. **5**, 296–300 (1954).

The author studies the zeros of functions of the form $F(x) = \int_x^\infty f(t)g(t)dt$ $(x>0)$ and obtains several results, of which the following are representative. Let $f(t) \geq 0$ on $[0, 1]$, $f \in L(0, 1)$, $f(t+n) = (-1)^n f(t)$, $f(t) \not\equiv 0$ on any interval. Suppose $g(t)$, as well as $g(t) - g(t+1)$, decreases to 0 monotonically as $t\to\infty$, and $g(n+1) < g(n)$ for infinitely many integers n. Choose c in $(0, 1)$ such that $2\int_0^c f(t)dt = \int_0^1 f(t)dt$. Then $F(x)$ has precisely one zero, z_n, in each interval $(n, n+1)$, and $z_n - n \geq c$. If also $g(n+1)/g(n) \to 1$ as $n\to\infty$, then $z_n - n \to c$, and $(-1)^n F(n)/g(n) \to \int_0^c f(t)dt$. *W. Rudin* (Rochester, N. Y.).

Nicolescu, Miron. La différentielle totale directe du second ordre. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 3 (1951), 507–516 (1952). (Romanian. Russian and French summaries)

The author considers various differentials, and derivatives, of the second order of a real function $u(x, y)$, and he considers some of their mutual properties. We shall quote two typical results. Let

$$\Delta_2 u \equiv u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) - u(x+\Delta x, y) + u(x, y),$$

$$\Delta_2^* u \equiv u(x+\Delta x, y+\Delta y) - u(x-\Delta x, y+\Delta y) - u(x+\Delta x, y-\Delta y) + u(x-\Delta x, y-\Delta y),$$

$$Du(x, y) \equiv \lim_{\Delta x\to 0, \Delta y\to 0} \operatorname{reg} \frac{\Delta_2 u(x, y)}{\Delta x \Delta y},$$

$$D^* u(x, y) \equiv \lim_{\Delta x\to 0, \Delta y\to 0} \operatorname{reg} \frac{\Delta_2^* u(x, y)}{4\Delta x \Delta y},$$

where these two latter limits are "regular limits". The author shows that if the Schwarz symmetric derivatives $\delta^2 u/\delta x^2$, $\delta^2 u/\delta y^2$, and Du are continuous near (x_0, y_0), then $D^* u(x_0, y_0) = Du(x_0, y_0)$. In addition, if $\delta^2 u/\delta x^2$, $\delta^2 u/\delta y^2$ exist near (x_0, y_0), and if $\delta^2(\delta^2 u/\delta x^2)/\delta^2 y$ is continuous near (x_0, y_0) then

$$\frac{\delta^2}{\delta x^2}\left(\frac{\delta^2 u}{\delta y^2}\right) = \frac{\delta^2}{\delta y^2}\left(\frac{\delta^2 u}{\delta x^2}\right)$$

at (x_0, y_0). *M. O. Reade* (Grenoble).

Conti, Roberto. Sulla convergenza in media delle derivate di una successione di funzioni convergente in lunghezza. Rend. Sem. Mat. Univ. Padova 23, 86–90 (1954).

The author demonstrates the absolute continuity of the limit function in a convergence theorem of E. Baiada [Ann. Scuola Norm. Super. Pisa (3) 6, 59–68 (1952); these Rev. 14, 628]. As a result, convergence in measure of the sequence of derivatives can be strengthened to convergence in the mean of order 1. *T. A. Botts* (Charlottesville, Va.).

Narain, R. D. On a Ridée function. Ganita 2, 1–8 (1951).

The author constructs a Ridée function. A Ridée function $f(x)$ is a function which is defined in a domain D of the real line, and which has the following properties: (a) $f(x)$ is continuous; (b) in every set E ($E \subset D$) of positive measure there exists a subset L, of zero measure, such that $f(x)$ is monotone on L and the image $f(L)$ of L is of positive measure. The first Ridée function was constructed by N. Bary [Math. Ann. 103, 185–248 (1930)], who proved (a) that every continuous function $F(x)$ can be represented in the form $F(x) = \sum_{i=1}^{3} f_i[\varphi_i(x)]$, where f_i and φ_i are absolutely continuous, and (b) that no Ridée function $F(x)$ can be represented in the form $F(x) = \sum_{i=1}^{2} f_i[\varphi_i(x)]$, where the f_i and φ_i are absolutely continuous. The author uses a method employed by A. N. Singh [Proc. Benares Math. Soc. (N.S.) 4, 95–108 (1943); these Rev. 5, 232]; the construction is unlike that given by N. Bary. *G. B. Price.*

Četković, Simon. La relation entre l'ordre des nombres algébriques et la différenciabilité d'une famille des fonctions. Bull. Soc. Math. Phys. Serbie 5, no. 3–4, 91–92 (1953). (Serbo-Croatian. French summary)

Après avoir défini une famille de fonctions de la manière suivante: $F(x, a) = 0$, si x est un nombre irrationnel, $F(x, a) = q^{-a}$, si x est un nombre rationnel de la forme p/q, p et q étant des entiers sans facteurs communs avec $q \neq 0$ et où a est un paramètre arbitraire, l'auteur démontre: La fonction $F(x, a)$ a une dérivée aux tous points irrationnels et algébriques de l'ordre $\leqq n$ pour $n < a$, quoique elle ne soit pas continue aux points d'un ensemble de points partout dense. *Résumé de l'auteur.*

Theory of Functions of Complex Variables

***Plemelj, Josip. Teorija analitičnih funkcij. [Theory of analytic functions.]** Slovenska Akademija Znanosti in Umetnosti, Ljubljana, 1953. xvi+516 pp.

This is a textbook of the theory of analytic functions of one complex variable in the best classical style. Besides the usual elementary topics it treats the elliptic functions, the elliptic modular functions, Picard's and Landau's theorems, potentials, the Dirichlet problem and mapping theorem, uniformization, and Abelian integrals. The book merits to be used widely. It is unfortunate that this is impossible because of the little known language (Slovenian) it is written in. *M. Golomb* (Lafayette, Ind.).

Smith, R. A. On an analytic function having an infinite number of independent real periods. J. London Math. Soc. 29, 255–256 (1954).

An example of a multiple-valued analytic function having an infinite number of independent real periods is constructed. The example in question is obtained by taking the inverse of a primitive of $\exp\{(1+i\pi)z\}/2i \sin \pi z$. *M. Heins* (Providence, R. I.).

Talanov, D. I. On some questions of the theory of iteration of a rational function. Doklady Akad. Nauk SSSR (N.S.) 93, 413–416 (1953). (Russian)

A study is made of the iterates $z_1, z_2, \cdots$ of an arbitrary point z_0 of the extended complex plane, generated by a given rational function $R(z)$ ($z_n = R(z_{n-1}) = R_n(z_0)$). A point ξ is a fixed point (of order n) if $\xi = \xi_n$ (but not $\xi = \xi_k$ for any k in $0<k<n$). The iterates of a fixed point ξ of order n form a cycle of order n, denoted by (ξ). Point z is a mobile point if all its iterates are distinct. A point is irregular if the family of functions $\{R_n(z)\}$ ($n=1, 2, \cdots$) is not normal in its neighborhood. The set of irregular mobile points has the power of the continuum; and the sequence of iterates of an irregular mobile point form a set dense in itself. Let (ξ) be a cycle of order n; then a point z is called a "dislodged" point relative to (ξ) if there exists a neighborhood U of ξ containing z, and an integer q, such that z_{kn} ($=kn$th iterate of z) is the highest iterate of z in U_{kn}, and is not in $U_{(k-1)n}$, where $k=1, 2, \cdots, q$. The greatest possible q is the degree of dislodgment of z. (Here U_j is the jth iterate of U.) A neighborhood of ξ is "dislodged" relative to (ξ) if each of its points is; and the union of all such neighborhoods of ξ is the region of dislodgment of ξ. It is shown that: (i) a repulsive fixed point always has a region of dislodgment; (ii) if a sequence of distinct repulsive fixed points converges, then the sequence of their regions of dislodgment converges to the point; (iii) a mobile irregular point is a point of dislodgment for an infinite set of repulsive cycles.

The remainder of the work deals with "partial limits" and "multiple convergence". An isolated limit point of a sequence is a partial limit of the first class, and a partial limit of pth class is a limit point of partial limits of arbitrary class $<p$. A sequence whose limits of pth class form a finite

set is a sequence of pth class. It is shown that a sequence of iterates is either of first class or of infinite class. A sequence of pth class is p-multiply convergent to z if z is the unique partial limit of class p. Given a repulsive fixed point ξ, there is an irregular mobile point z such that for each p some subsequence of iterates of z is p-multiply convergent to ξ.

I. M. Sheffer (State College, Pa.).

Sunyer Balaguer, F. Approximation of functions by sums of exponentials. Collectanea Math. **5**, 241–267 (1952). (Spanish)

The author extends the "fundamental inequality" of Mandelbrojt [Séries adhérentes, $\cdots$, Gauthier-Villars, Paris, 1952; these Rev. **14**, 542] and applies his results to the representation of an analytic function as a limit of exponential polynomials. In the definition of the representation of a function with given logarithmic precision, the approximating functions no longer have to be the partial sums of a Dirichlet series, but may be any sequence of exponential polynomials with exponents chosen from a prescribed sequence. Under hypotheses similar to Mandelbrojt's the author then shows that a function which is represented in a suitable domain with an appropriate logarithmic precision is the limit of a sequence of exponential polynomials with exponents from the same sequence. Some of Mandelbrojt's uniqueness theorems are then generalized. The author considers the effect of introducing a generalization of logarithmic precision [cf. C. R. Acad. Sci. Paris **232**, 669–671 (1951); these Rev. **12**, 489]. He generalizes still further by allowing the exponents to be complex. Finally he gives some results on the question of which functions are representable with a given logarithmic precision by exponential polynomials with given exponents.

R. P. Boas, Jr.

Makar, Ragy H. On algebraic basic sets of polynomials. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 57–68, 69–76 (1954).

Makar, Ragy H. Effect of the addition of the unit set on the order of a simple monic set of polynomials. Duke Math. J. **21**, 75–78 (1954).

The author continues to explore the effect on a basic set of polynomials $\{p_n(z)\}$ of the hypothesis that it is algebraic, i.e. that its coefficient matrix satisfies an algebraic equation. In the first paper the condition

$$A(R) = \limsup_{n\to\infty} \{\max_{|z|=R} |p_n(z)|\}^{1/n} \leqq R, \quad a \leqq R < b,$$

is shown to imply that an algebraic basic set is effective in $|z| \leqq R, a \leqq R < b$. The condition $A(r) < R$ for $r < R$ implies the effectiveness in $|z| < R$ of an algebraic basic set. The condition $A(0+) = 0$ implies the effectiveness at 0 of an algebraic basic set. Examples show that the condition that the set is algebraic is essential. The condition $A(0+) < \infty$ does not imply the effectiveness of an algebraic basic set at 0 for all entire functions, but with the condition $\limsup n^{-1} \deg (p_n) < \infty$ it does.

The second paper discusses the effectiveness of the set whose matrix is a polynomial in the matrix of a given algebraic simple monic set, of sets whose zeros are in a fixed circle, and of sets whose coefficients satisfy certain inequalities.

The third paper shows that for an algebraic simple monic set of degree n and order ω, the order Ω of $a\{p_n(z)\} + b\{z^n\}$, $a + b = 1$, satisfies $\omega/(m-1) \leqq \Omega \leqq \omega(m-1)$, both bounds being attainable.

R. P. Boas, Jr. (Evanston, Ill.).

Makar, Ragy H. On derived and integral basic sets of polynomials. Proc. Amer. Math. Soc. **5**, 218–225 (1954).

The author effectively disposes of the problem of determining the convergence properties of the basic polynomial set obtained by integrating or differentiating another such set (at least, within the framework of the Whittaker theory). He gives a systematic presentation of old and new results. The principal facts are that the integrated set always has at least as good convergence properties as the original set; for the differentiated set this is true except in certain cases, when auxiliary conditions are required. The chief new result on differentiated sets is that if $\lim (\log D_n)/(n \log n) = 0$ (D_n is the degree of the polynomial of highest degree occurring in the representation of z^n), but not necessarily otherwise, then if the basic set is effective in $|z| \leqq R$ for every entire function of order less than ρ, the differentiated set has the same property.

R. P. Boas, Jr. (Evanston, Ill.).

Denjoy, Arnaud. L'expression asymptotique des fonctions entières. C. R. Acad. Sci. Paris **238**, 1077–1080 (1954).

Consider an entire function $R(u)$ of order less than 1 with real zeros $-\rho_n = \rho(n)$, where ρ is an analytic function vanishing at 0 and real and increasing on the positive real axis. Let $S(u) = R'(u)/R(u) = \sum 1/(u + \rho(n))$; the author compares $S(u)$ with $V(u) = \int_0^\infty (u + \rho(n))^{-1} dn$, finding

$$S(u) = V(u) - \tfrac{1}{2}u^{-1} + \Delta(u),$$

where $\Delta(u)$ has an asymptotic expansion in negative powers of u which can be given explicitly in simple cases, e.g. for $\rho(n) = n^{1/\theta}$, $\theta < 1$.

R. P. Boas, Jr. (Evanston, Ill.).

Ahmad, Mansoor. A note on a theorem of Borel. Math. Student **21** (1953), 105–106 (1954).

The author proves the following special case of a result of Borel [Leçons sur les fonctions entières, 2ième éd., Gauthier-Villars, Paris, 1921, pp. 96–97]. For any entire function $f(z)$ of integral order $\rho \geqq 1$ there exists at most one entire function $f_1(z)$ of order less than ρ such that the zeros of $f(z) - f_1(z)$ have exponent of convergence less than ρ. The proof is similar to that of Borel.

J. Korevaar.

Schild, A. On a class of functions schlicht in the unit circle. Proc. Amer. Math. Soc. **5**, 115–120 (1954).

The class S_p of polynomials $f_p(z) = z - \sum_2^N a_n z^n$, $a_n \geqq 0$, is considered that have the circle $|z| < 1$ as exact circle of schlichtness. By very elementary arguments quite interesting properties of S_p are obtained. For instance: (i) $\sum_2^N n a_n = 1$ is necessary and sufficient for $f_p \in S_p$. (ii) The map of $|z| \leqq 1$ by f_p always covers $|w| \geqq \frac{1}{2}$ and is starlike with respect to $w = 0$, but never convex. (iii) The map of $|z| \leqq \frac{1}{2}$ is always convex. (iv) $|z| - \frac{1}{2}|z|^2 \leqq |f_p(z)| \leqq |z| + \frac{1}{2}|z|^2$ and $1 - |z| \leqq |f_p'(z)| \leqq 1 + |z|$ when $|z| \leqq 1$. All these results are best possible.

W. W. Rogosinski.

Jenkins, James A. A recent note of Kolbina. Duke Math. J. **21**, 155–162 (1954).

Suppose $f_1(z)$, $f_2(z)$ are schlicht functionsm apping $|z| < 1$ onto mutually exclusive domains, such that $f_1(0) = a_1$, $f_2(0) = a_2$. Let α, β be given positive numbers. Then the extremum value of $|f_1'(0)|^\alpha |f_2'(0)|^\beta$ is

$$4^{\alpha+\beta} \frac{\alpha^\alpha \beta^\beta}{|\alpha - \beta|^{\alpha+\beta}} \left| \frac{\alpha^{1/2} - \beta^{1/2}}{\alpha^{1/2} + \beta^{1/2}} \right|^{2\alpha^{1/2}\beta^{1/2}} |a_1 - a_2|^{\alpha+\beta}.$$

Some related extremum problems are also solved. The author's method depends on extremum lengths and quad-

ratic differentials. Previous solutions of these problems due to Kolbina [Doklady Akad. Nauk SSSR (N.S.) **84**, 865–868 (1952); these Rev. **14**, 35] and Kufarev and Fales [ibid. **81**, 995–998 (1951); these Rev. **14**, 262] depended on Golusin's variational method and Löwner's differential equation respectively. *W. K. Hayman* (Exeter).

Plotnick, Samuel I., and Benton, Thomas C. Evaluation of constants in conformal representation. Quart. Appl. Math. **12**, 76–77 (1954).

The authors evaluate a multiplicative constant contained in the Schwarz-Christoffel formula on polygonal mapping in a special case where the polygon possesses a half-strip with a given breadth. *Y. Komatu* (Tokyo).

Meschkowski, Herbert. Verallgemeinerung der Poissonschen Integralformel auf mehrfach zusammenhängende Bereiche. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **166**, 12 pp. (1954).

Let B be a multiply connected plane region whose boundary C consists of smooth Jordan arcs. Let $r(z;u,v)$ and $k(z;u,v)$ be the functions which map B onto the regions bounded by radial and circular slits with u going into 0 and v into ∞. Set $Q(z;u,v)=\frac{1}{2}[\log k(z;u,v)-\log r(z;u,v)]$. Then for any function f analytic in $B+C$

$$f(u)-f(v)=\frac{1}{\pi i}\int_C (Rf)Q'(z;u,v)dz.$$

H. L. Royden (Stanford, Calif.).

Kubo, Tadao. Bergman kernel function and canonical slit-mapping. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **28**, 33–40 (1953).

Let D be a domain the complex z-plane containing the points $z=0$, ∞ and let $w=f_\theta(z)$ be the function, normalized by $f_\theta(0)=0$, $f_\theta'(0)=1$, which maps D onto a schlicht domain bounded by spiral slits Re $\{e^{-i\theta}\log w\}=$const. If A denotes the logarithmic area of the complement of D and $f_\theta(z)$ has the expansion $f_\theta(z)=z+b+a_\theta z^{-1}+\cdots$ near $z=\infty$, it is proved that

$$\mathrm{Re}\,\{e^{-i}\log a_\theta\}-|\log a_\theta|^2[\log a_{\pi/2}-\log a_0]^{-1}\geqq(2\pi)^{-1}A.$$

The author is obviously not aware of the fact that the corresponding result for parallel slit mappings has been proved by B. Epstein [Bull. Amer. Math. Soc. **53**, 813–819 (1947); these Rev. **9**, 180] by precisely the same method. *Z. Nehari* (Pittsburgh, Pa.).

Jacobsthal, Ernst. Über die Klasseninvariante ähnlicher linearer Abbildungen. II. Norske Vid. Selsk. Forh., Trondheim **26**, 10–15 (1953).

A continuation of earlier studies on linear fractional conformal transformations [same Forh. **25**, 119–124 (1953); these Rev. **14**, 968]. *P. R. Garabedian.*

Royden, H. L. A property of quasi-conformal mapping. Proc. Amer. Math. Soc. **5**, 266–269 (1954).

Let O_G and O_{HD} be the classes of Riemann surfaces with G- and HD-removable boundaries. Consider two open Riemann surfaces W_1 and W_2 such that there is a quasi-conformal mapping of W_1 onto W_2. The author shows that the ring of bounded harmonic functions with a finite Dirichlet integral is the same for both W_1 and W_2. From this he deduces that (1) the class O_G is preserved under quasi-conformal mapping [Pfluger, C. R. Acad. Sci. Paris **227**, 25–26 (1948); these Rev. **10**, 28], (2) the same is true of O_{HD}. *L. Sario* (Cambridge, Mass.).

Huckemann, Friedrich. Bestimmung der Wertverteilung der Gammafunktion aus ihrer Riemannschen Fläche. Math. Z. **59**, 375–382 (1934).

Huckemann, Friedrich. Typusänderung bei Riemannschen Flächen durch Verschiebung von Windungspunkten. Math. Z. **59**, 383–387 (1954).

In the first paper it is shown that the growth and value distribution of the gamma-function can be derived, to a certain degree of approximation, from the geometric properties of the corresponding Riemann surface.

In the second paper an example is constructed to illustrate that the type of a Riemann surface may change when the branch-points are moved. The author considers the surface with simple algebraic branch-points at -1, 0, 1 and logarithmic branch-points at ∞, in all the sheets. If the branch-points over ±1 are moved to $\pm p_n$, it is shown that the surface remains hyperbolic if the p_n tend slowly to ∞, and becomes parabolic if the convergence is sufficiently rapid. *L. V. Ahlfors* (Cambridge, Mass.).

***Mehring, Johannes. Kernfunktion und Regularitätsgebiete im Raum von zwei komplexen Veränderlichen.** Dissertationen der mathematisch-naturwissenschaftlichen Fakultät der westfälischen Wilhelms-Universität zu Münster in Referaten, Heft 4, pp. 5–7. Aschendorffsche Verlagsbuchhandlung, Münster, 1954.

***Stein, Karl. Analytische Projektion komplexer Mannigfaltigkeiten.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 97–107. Georges Thone, Liège; Masson & C^ie, Paris, 1953.

Have

Soit $f=f(z_1,\cdots,z_n)$ une fonction méromorphe non constante et sans point d'indétermination sur une variété analytique complexe M^{2n}; par projection analytique notée $R^2(M^{2n};f)$ de la variété M^{2n} selon f on entend une relation d'équivalence entre les points de M^{2n} et une fonction à valeurs complexes $w=W(p)$ définie sur les classes d'équivalence (p), dont on donnera une définition de caractère constructif. Aux points de (p), f prend la même valeur. La relation $(P_1,P_2;Q_1,Q_2)|(M^{2n};f)$ signifie qu'il existe deux arcs de Jordan P_1Q_1, P_2Q_2 sur M^{2n} évitant les points non ordinaires (où toutes les dérivées $f^{(k)}=\partial f/\partial z_k$ s'annulent) et tels que les points $M(t)$, $M'(t)$, $0\leqq t\leqq1$, parcourent P_1Q_1 et P_2Q_2 de manière que $f[M(t)]=f[M'(t)]$. Alors $R^2(M^{2n};f)$ est la classification la plus fine sur M^{2n} qui vérifie: si P_1 et P_2 appartiennent à une même classe K_i, Q_1 et Q_2 appartiennent aussi à une même classe K_j. La construction des classes peut être obtenue par les opérations suivantes: (1) Q et Q' sont dits $(f,0)$ équivalents s'ils coïncident; (2) Q_1 et Q_2 sont (f,k) associés s'ils sont $(f,k-1)$ équivalents ou s'il existe deux points P_1 et P_2 qui sont $(f,k-1)$ équivalents, la relation $(P_1,P_2;Q_1,Q_2)|(M^{2n};f)$ étant satisfaite; Q et Q' sont dits (f,k) équivalents si l'on passe de Q à Q' par une suite finie de points de M^{2n} dont deux consécutifs sont (f,k) associés; deux points sont f équivalents ou dans la même classe s'ils sont (f,k) équivalents pour une valeur de k finie. Un ensemble A sur l'ensemble $\mathfrak{R}$ des classes (p) est ouvert s'il existe un ouvert A' de M^{2n} qui est représenté sur A dans le passage φ de $\mathfrak{R}$ à M. $R^2(M^{2n};f)$ est une surface de Riemann sur le plan de la variable w; φ est analytique complexe et $f=W\cdot\Phi$.

On dira que f_1 méromorphe sur M^{2n} est dépendante de f si en tout point de M^{2n} ordinaire pour f et f_1 la matrice

$(f^{(k)}, f_1^{(k)})$ est de rang inférieur à deux: on établit qu'il existe alors un homéomorphisme de $R^2(M^{2n}; f)$ sur $R^2(M^{2n}; f_1)$ [voir la Thèse de K. Koch, Wilhelms-Univ., Münster, 1952; ces Rev. 15, 25].

La représentation Φ fournit un homorphisme du groupe fondamental $\pi_1(M^{2n})$ sur $\pi_1[R(M^{2n}; f)]$, lié au fait que deux points équivalents Q et Q' peuvent être reliés par un arc dont l'image par Φ peut-être réduite par déformation homotopique dans $R^2(M^{2n}; f)$ à $p=\Phi(Q)=\Phi(Q')$.

Si M^{2n} est un domaine D de $C^n(z_k)$ et $H(D)$ sa cellule de régularité, alors on a $R(D; f)=R[H(D); f]$. Le résultat suivant montre la portée et l'efficacité de la méthode de la projection analytique; que M^{2n} soit compacte ou non, f étant méromorphe sur M^{2n} avec un ensemble V non vide de points d'indétermination, la projection analytique $R^2(M^{2n}-V; f)$ est compacte. Il en résulte que si deux fonctions f_1 et f_2 méromorphes sont dépendantes sur M^{2n}, leur dépendance est nécessairement algébrique si l'une au moins a des points d'indétermination (résultat obtenu déjà d'une manière un peu moins précise et par une méthode plus compliquée par W. Thimm [Math. Ann. 125, 145–164 (1952); 264–283 (1953); ces Rev. 15, 210]), ou si M^{2n} est une variété compacte. *P. Lelong* (Lille).

***Behnke, H. Généralisation du théorème de Runge pour des fonctions multiformes de variables complexes.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 81–96. Georges Thone, Liège; Masson & C^ie, Paris, 1953.

Have

On étudie, étant donné un domaine G, à quelles conditions une fonction holomorphe quelconque dans G peut être approchée dans G (au sens de la convergence uniforme sur tout compact de G) par des fonctions holomorphes dans le domaine $G'\supset G$. Dans le cas des fonctions d'une seule variable complexe on a l'extension suivante du théorème de Runge: G et G' étant des domaines sur une surface de Riemann non fermée R, pour que le problème posé soit possible, il faut et il suffit que G soit simplement connexe par rapport à G', ce qui équivaut à: tout cycle dans G homologue à zéro dans G' l'est encore dans G, ou encore à la possibilité d'une extension semi-continue de G à G' par des domaines $G(t)$, $0\leq t\leq 1$, $G(0)=G$, $G(1)=G'$: la démonstration s'appuie sur l'extension de l'intégrale de Cauchy donnée par Behnke et Stein [Math. Ann. 120, 430–461 (1949), p. 437; ces Rev. 10, 696].

L'extension aux fonctions de n variables amène à préciser ce qu'on entendra par une extension régulière du domaine G à un domaine G' dans l'espace C^n; on posera: (1) $G\subset G'$, (2) pour tout domaine $G_0\subset\subset G$, pour $G_0'\subset\subset G'$ et pour tout $\epsilon>0$, il existe une chaîne $G_1, \cdots, G_s$ de domaines d'holomorphie avec (a) $G_0\subset\subset G_1\subset\subset G$, (b) $G_0'\subset\subset G_s\subset\subset G'$, (c) $G_0\subset G_j^\epsilon$, $j=1, \cdots, s$, (d) la composante de G_j^ϵ contenant G_0 est dans G_{j-1}; G^ϵ désigne l'ensemble des points de G dont la distance à la frontière de G surpasse ϵ. On a alors: pour que le problème posé soit possible, il faut et il suffit que l'on puisse agrandir G en G' par une extension régulière.

Le résultat s'étend à des domaines G et G' pris sur une variété M^{2n} à structure analytique complexe en supposant que G est faiblement holomorphe convexe (en d'autres termes que G est une variété de Stein) sur M^{2n}: l'extension régulière de G en G' utilise une métrique, mais les conditions (c) et (d) sont indépendantes de cette métrique supposée continue et bornée sur G et sa frontières. On dira que le domaine $G\subset M^{2n}$ est fortement holomorphe convexe par rapport à la métrique $(\mathfrak{M})$ s'il est borné pour $(\mathfrak{M})$, s'il est faiblement holomorphe convexe et si l'enveloppe faiblement holomorphe convexe d'un compact intérieur à G peut toujours être prise dans G^ϵ, pour un choix de ϵ assez petit.

On a alors l'énoncé: soit G' une variété de Stein sur M^{2n} et soit P un polyèdre analytique sur G': on peut trouver une métrique continue rendant fortement convexe toute variété de Stein contenue dans P. Pour que le problème posé au début soit possible, G et G' étant des variétés de Stein sur M^{2n}, il faut et il suffit qu'il existe une extension régulière de G à G'; cette possibilité d'extension régulière constitue une condition nécessaire et suffisante pour que G soit convexe par rapport aux fonctions holomorphes dans G' [voir pour les démonstrations détaillées de ces derniers résultats la thèse de H. Will, Wilhelms-Univ., Münster, 1952; ces Rev. 15, 25]. *P. Lelong* (Lille).

Poor, Vincent C. On residues of polygenic functions. Trans. Amer. Math. Soc. 75, 244–255 (1953).

In this paper the author generalizes the idea of the residue of analytic functions, over an area, to polygenic functions. These results extend the author's earlier results [same Trans. 32, 216–222 (1930)]. The author defines the residue of polygenic $f(z)$ over the domain σ bounded by the simple, closed, rectifiable curve γ to be

$$R'=\frac{1}{2\pi i}\int_\gamma f(z)dz-\frac{1}{\pi}\int\int_\sigma Af(z)d\sigma,$$

where $Af(z)=\frac{1}{2}(\partial u/\partial x-\partial v/\partial y)+\frac{1}{2}i(\partial u/\partial y+\partial v/\partial x)$ is the areal derivative of $f(z)$. He then shows that if $f(z)$ is polygenic in σ and if $f(z)$ has a simple pole at $z=a$ in σ, and if $g(z)\equiv(z-a)f(z)$, then the residue R' of $f(z)$ over σ is given by $R_\sigma(f)=g(a)$. The author defines a second residue

$$\bar{R}'=-\frac{1}{2\pi i}\int_\gamma f(z)d\bar{z}-\frac{1}{\pi}\int\int_\sigma A^*f(z)d\sigma,$$

where $A^*f(z)$ is the "mean derivative" of $f(z)$. Then for the $f(z)$, polygenic in σ and with simple pole at $z=a$, with $g(z)\equiv(z-a)f(z)$, we have $\bar{R}_\sigma'(f)=0$. The author then defines a total residue $R=R'+\bar{R}'$. He extends the preceding results to the cases of polygenic functions with a finite number of poles, not necessarily simple, to functions of the form $[f(z)/(az+b\bar{z})]$, and to functions of the form $[f(z)/(az^2+bz\bar{z}+c\bar{z}^2)]$. *M. O. Reade.*

de Mira Fernandes, A. Funzioni pseudo-monogenee. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. 2, 77–88 (1952).

Let $f(z, z')$ be a function of two complex variables, and define the operators

$$L=\frac{\partial^2}{\partial x\partial x'}+\frac{\partial^2}{\partial y\partial y'},\quad L'=\frac{\partial^2}{\partial x'\partial y}-\frac{\partial^2}{\partial x\partial y'},$$

$$\Lambda=L+iL',\quad D=\tfrac{1}{2}(L-iL').$$

$f(z, z')$ is said to be pseudo-monogenic if and only if $\Lambda f=0$. In this paper, the author shows that if $f(z, z')$ is pseudo-monogenic, then the "derivative" Df is also pseudo-monogenic, and hence D^nf exists for each positive integer n. The author also considers pseudo-harmonic functions $u(x, y, x', y')$ satisfying $(L^2+L'^2)u=0$. For these latter functions, he obtains mean-value theorems. *M. O. Reade* (Grenoble).

Fréchet, Maurice. Determination of the most general plane para-analytic function. Ann. Mat. Pura Appl. (4) 35, 255–268 (1953). (Esperanto. French summary)

Let $v=xe_1+ye_2$ and $V=Xe_1'+Ye_2'$ be two plane vectors, and suppose the basis vectors satisfy a multiplication rule: $e_k'e_j=\sum u_{kjh}e_h'$. Then V is called a "para-analytic" function of v with respect to the given multiplication rule if X and Y are differentiable with respect to x and y and there is a vector $Le_1'+Me_2'$ such that

$$dXe_1'+dYe_2'=(Le_1'+Me_2')(dxe_1+dye_2).$$

The author shows that every such para-analytic function must be one of five types. *H. L. Royden.*

Fréchet, Maurice. Sur les surfaces dérivables relativement à une règle de multiplication hypercomplexe. C. R. Acad. Sci. Paris 238, 633–636 (1954).

The author solves the problem of finding all para-analytic mappings from the plane into three-dimensional space. [See the preceding review.] *H. L. Royden* (Stanford, Calif.).

Theory of Series

Ramanujan, M. S. On summability methods of type *M*. J. London Math. Soc. 29, 184–189 (1954).

The author proves several theorems on matrices A of regular matrix transformations of the types (i) sequence-to-sequence, (ii) series-to-sequence, and (iii) series-to-series. The theorems give conditions which are necessary or sufficient or both to ensure that A be of type M. A matrix A of type (i) has type M if the two conditions $\sum|c_n|<\infty$ and $\sum c_k a_{nk}=0$ when $n=1, 2, \cdots$ imply that $c_k=0$ for each k. For matrices of types (ii) and (iii), similar definitions of type M are given. In some cases, the matrix A in question is the product BC of matrices about which hypotheses are given. In other cases, A is a factor in the product AB and hypotheses on AB and B are given. *R. P. Agnew.*

Al'tman, M. Generalization of a theorem of Mazur-Orlicz from the theory of summability. Studia Math. 13, 233–243 (1953). (Russian)

Let A be a method by which a sequence $s_0, s_1, \cdots$ is evaluable to s if the series in $A(t)=\sum a_k(t)s_k$ converges for each t in T and $A(t)\rightarrow s$ as t approaches a limit point t_0 of s. Let B be another method of this type. Let A and B evaluate to zero each sequence which converges to zero. Let B evaluate each bounded sequence which A evaluates to zero. Suppose, moreover, that A satisfies a particular additional condition which is too complicated for presentation here. Then B evaluates to zero each bounded sequence which is evaluable A to zero. *R. P. Agnew* (Ithaca, N. Y.).

Ogieveckiĭ, I. E. On the comparability of the Abel and (C, α, β) methods of summation. Doklady Akad. Nauk SSSR (N.S.) 92, 231–234 (1953). (Russian)

If $\alpha, \beta>-1$ and if $\sum U_{nk}$ is a double series which is restrictedly evaluable to S by the Abel method and which has a $C(\alpha, \beta)$ transform satisfying a stated set of three conditions, then $\sum U_{nk}$ is restrictedly evaluable $C(\alpha+1, \beta+1)$. Two more similar theorems are given. *R. P. Agnew.*

Tatchell, J. B. A note on a theorem by Bosanquet. J. London Math. Soc. 29, 207–211 (1954).

The author proves a factor theorem for absolute Abel evaluability of series. In order that a factor sequence ϵ_n be such that the series $\sum a_n\epsilon_n$ is evaluable $|A|$ whenever $\sum a_n$ is convergent, it is necessary and sufficient that $\sum|\Delta\epsilon_n|<\infty$ and $\sum_{n=1}^{\infty} n^{-1}|\epsilon_n|<\infty$. This theorem and results of Bosanquet [same J. 20, 39–48 (1945); these Rev. 7, 432] imply the following. If k is a nonnegative integer and $r>k+1$, then necessary and sufficient conditions that $\sum a_n\epsilon_n$ be evaluable $|C, r|$ whenever $\sum a_n$ is evaluable (C, k) are $\sum n^k|\Delta^{k+1}\epsilon_n|<\infty$ and $\sum n^{-1}|\epsilon_n|<\infty$. *R. P. Agnew.*

Rauch, S. E. Mapping properties of Cesàro sums of order two of the geometric series. Pacific J. Math. 4, 109–121 (1954).

Let

$$S_n^{(2)}(e^{i\phi})=e^{in\phi}+\binom{3}{2}e^{i(n-1)\phi}+\cdots+\binom{n+2}{2}=x_n(\phi)+iy_n(\phi)$$

be the nth Cesàro sum of order 2 for the geometrical series at $z=e^{i\phi}$. The asymptotic behaviour of $x_n(\phi)$ and $y_n(\phi)$ is studied. Of the results obtained the following one is characteristic: for large n, y_n increases for $0<\phi<a_n$ and decreases for $a<\phi<\pi$ where $a_n=a/n+O(n^{-2})$ and $\pi<a<3\pi/2$.
W. W. Rogosinski (Newcastle-upon-Tyne).

Basu, S. K. On hypergeometric summability involving infinite limits. Proc. Amer. Math. Soc. 5, 226–238 (1954).

The hypergeometric summability $(H, \alpha, \beta, \gamma)$ (where $\alpha, \beta, \gamma>0$) is a special type of Hausdorff summability and related with the Cesàro methods [see Agnew, Amer. J. Math. 63, 705–708 (1941); these Rev. 3, 149]. The method is regular for $\gamma\geqq\alpha+\beta-1$. The author shows in theorems 2–4 that $(H, \alpha, \beta, \gamma)$ is totally regular if (i) $\gamma>\alpha\beta>\alpha+\beta-1$, $0<\alpha, \beta<1$ and $\gamma>\alpha$ or β, or if (ii) $\gamma\geqq\alpha+\beta-1>\alpha\beta$, and that $(H, \alpha, \beta, \gamma)$ is not totally regular if (iii) $\gamma>\alpha\beta>\alpha+\beta-1$ and $\alpha, \beta>1$ or if (iv) $\alpha+\beta-1\leqq\gamma\leqq\alpha\beta$ and $\alpha, \beta\neq1$. Whether $(H, \alpha, \beta, \gamma)$ is totally regular whenever (v) $\gamma>\alpha\beta>\alpha+\beta-1$ and $0<\alpha, \beta<1$ remains an open question. The cases when α or $\beta=1$ are trivial. The proofs are based on theorem 1 which states that $C_\alpha C_\beta$ (Cesàro methods) is totally stronger than $C_{\alpha+\beta}$ if and only if $\alpha\beta\geqq0$ and that $C_{\alpha+\beta}$ is totally stronger than $C_\alpha C_\beta$ if and only if $\alpha\beta\leqq0$ $(\alpha, \beta, \alpha+\beta>-1)$. [Cf. previous work of the author, Proc. London Math. Soc. (2) 50, 447–462 (1949); J. London Math. Soc. 24, 51–59 (1949); these Rev. 10, 368, 447.] *K. Zeller.*

Basu, S. K. On the total relative strength of some Hausdorff methods equivalent to identity. Amer. J. Math. 76, 389–398 (1954).

Es sei H eine total-reguläre Hausdorff-Matrix [d.h. (1) aus $\lim s_n=s$ folgt H-$\lim s_n=s$ für $-\infty\leqq s\leqq+\infty$], ihre Momentfunktion sei

$$\mu(z)=\int_0^1 t^z d\chi(t) \quad (z=x+iy),$$

und $\chi(t)$ gehöre in $\langle 0, 1\rangle$ der Klasse V^* an [vgl. H. R. Pitt, Proc. Cambridge Philos. Soc. 34, 510–520 (1938), insbes. S. 511]. Verf. bildet für reelles α die reguläre Hausdorff-Matrix $U_\alpha=\alpha E+(1-\alpha)H$ (E=Einheitsmatrix) und vergleicht ihre Stärke mit E (Mercersche Sätze) und mit $U_\beta(\beta\neq\alpha)$. Satz 1. Es gilt: (2) $U_\alpha\approx E$ $(\alpha>\frac{1}{2})$; $U_\alpha\approx E$ $(0<\alpha\leqq\frac{1}{2})$, falls $\Re\mu(z)\geqq0$ in $x\geqq0$ ist; für $\alpha<0$ ist $U_\alpha\supset E$ (U_α stärker als E), aber nicht umgekehrt, falls $\chi(t)$ noch an $t=1$ stetig ist. Satz 2 behandelt totale Regularität von U_α;

so ist z.B. U_α total stärker als E [d.h. es gilt (1)], aber nicht umgekehrt $(0<\alpha<1)$. In den Sätzen 3 bis 6 werden zu demselben H gehörige U_α und U_β $(0<\beta<\alpha)$ verglichen. Z.B. gilt in $1<\beta<\alpha$ wegen (2) stets $U_\alpha \approx U_\beta$, und weiter ist U_β total stärker als U_α, aber nicht umgekehrt.

D. Gaier (Cambridge, Mass.).

Obreškov, Nikola. On some theorems on summation of divergent series. Bŭlgar. Akad. Nauk. Izvestiya Mat. Inst. 1, 3–26 (1953). (Bulgarian. Russian summary)

Let $\phi(x)$ be a positive function such that, for some real number m and each positive number λ,

$$\lim_{x\to\infty} [\phi(\lambda x)/\phi(x)] = \lambda^m.$$

Suppose that $0<x_1<x_1'<x_2<x_2'<\cdots$ and that, for some $\theta>0$, $x_k'-x_k>\theta x_k$. Let $f(x)$ be a real function for which $f^{(n)}(x)$ exists when $x_k<x<x_k'$, and for which $f(x)\sim A\phi(x)$ and $f^{(n)}(x)>-Mx^{-n}\phi(x)$ in these intervals. Suppose that $\phi(x)$ is, for each k, either monotone increasing or monotone decreasing over $x_k<x<x_k'$. Then, for each sufficiently small positive ϵ, and for each i in the interval $1\leqq i\leqq n-1$,

$$f^{(i)}(x)\sim Am(m-1)\cdots(m-i-1)x^{-i}\phi(x)$$

when $x\to\infty$ over the intervals $x_k(1+\epsilon)\leqq x\leqq x_k'(1-\epsilon)$. Three more similar theorems on derivatives of functions and differences of sequences are given. *R. P. Agnew.*

Orloff, Constantin. Transformations géométriques des séries à termes positifs. Bull. Soc. Math. Phys. Serbie 5_1 no. 3–4, 53–59 (1953). (Serbo-Croatian summary)

Some points in the theory of series $\sum a_k$ of positive terms are illustrated by use of plane figures in which the values of the terms of the series are lengths of horizontal line segments having their endpoints on plane curves. *R. P. Agnew.*

Korevaar, Jacob. A very general form of Littlewood's theorem. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 36–45 (1954).

The author proves a Tauberian theorem which we give with omission of some details. Let $\phi(t)=t^\alpha L(t)$, where $L(t)$ is slowly oscillating. Let $\omega(u)$ be a bounded positive function which converges monotonely to 0 as $u\to 0+$. Let $\sum a_n$ be a real series for which $a_n>-\phi(n)$ and

$$(1)\qquad \left|\sum_{n=0}^{\infty} a_n e^{-nu}-s\right|<\omega(u)$$

when $u>0$. Then, when $n>0$, $|\sum_{k=0}^{n} a_k - s|<\rho(n)$, where

$$\rho(n)=\min_{p\leqq K_1}\left\{K_2\frac{n\phi(n)}{p}+K_3 p\omega\left(\frac{p}{n}\right)\right\}$$

if $\liminf_{u\to 0} u\log\omega(u)>-\infty$ and $\rho(n)=0$ otherwise. The proof depends upon lemmas on polynomial approximation, principally one due to Freud [Acta Math. Acad. Sci. Hungar. 2, 299–307 (1953); these Rev. 14, 958], of which independent proofs are given. There are numerous examples.

R. P. Agnew (Ithaca, N. Y.).

Korevaar, Jacob. Another numerical Tauberian theorem for power series. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 46–56 (1954).

The principal theorem of this paper differs from that of the preceding in that the hypothesis (1) of the preceding review is replaced by a hypothesis of the form

$$\left|\sum_{n=0}^{\infty} a_n e^{-nu}-G(u)\right|\leqq\omega(u).$$

This theorem implies a simpler theorem to the effect that if $\sum a_n z^n$ converges when $|z|<1$ to a function $f(z)$ analytic at $z=1$, and if $a_n\geqq-\phi(n)$, where $\phi(n)=n^\alpha L(n)$ and $L(t)$ is slowly oscillating, then $|s_n-f(t)|\leqq C\phi(n)$.

R. P. Agnew (Ithaca, N. Y.).

Vučković, Vladeta. Une extension de la condition de convergence dans les théorèmes de nature taubérienne. Srpska Akad. Nauka. Zbornik Radova 35. Mat. Inst. 3, 75–84 (1953). (Serbo-Croatian. French summary)

Let $\alpha(x)$ be nondecreasing and continuous on the right on $(0,\infty)$, with $\alpha(\infty)=\infty$. Then $\int_0^x f(t)d\alpha(t)=o(1)$ $(x\to\infty)$ implies $f(x)=o(1)$ $(x\to\infty)$ if an appropriate Tauberian condition is satisfied. With a suitable $X=X(x,\epsilon)$, Schmidt's condition is

$$\liminf_{x\to\infty}\min_{x\leqq x'\leqq X}\{f(x')-f(x)\}\geqq-o(1),\quad \epsilon\to 0;$$

while Karamata has considered the two generalizations (with appropriate functions ρ, Φ)

$$\liminf_{x\to\infty}\min_{x\leqq x'\leqq X}\frac{\rho(x')f(x')-\rho(x)f(x)}{\rho(x)}\geqq-o(1),$$

$$\liminf_{x\to\infty}\min_{x\leqq x'\leqq X}\{\Phi[f(x')]-\Phi[f(x)]\}\geqq-o(1).$$

The author establishes the following Tauberian condition which generalizes both of Karamata's:

$$\liminf_{x\to\infty}\min_{x\leqq x'\leqq X}\frac{\rho(x')\Phi[f(x')]-\rho(x)\Phi[f(x)]}{\rho(x)}\geqq-o(1).$$

Here $\phi(x)$ is the left-continuous inverse of $\alpha(x)$ and $X=\phi[\alpha(x)+\epsilon]$; $\Phi(x)$ is continuous and strictly increasing; $\rho(x')\asymp\rho(x)$ uniformly for $x\leqq x'\leqq X$, and

$$\limsup_{x\to\infty}\max_{x\leqq x'\leqq X}\left|\frac{\rho(x')-\rho(x)}{\rho(x)}\right|=o(1),\quad\epsilon\to 0.$$

Special cases include the following. (1) If $F(x)\to A$ and $c\Phi[F'(x)]+\Phi'[F'(x)]F''(x)>O(1)$, then $F'(x)\to 0$. (2) If $\sum U_n$ converges and

$$U_{n+1}^{1/(2k+1)}-U_n^{1/(2k+1)}<(\mu/n)U_n^{1/(2k+1)}+Mn^{-1-1/(2k+1)},$$

then $nU_n\to 0$. *R. P. Boas, Jr.* (Evanston, Ill.).

Borwein, D. Note on summability factors. J. London Math. Soc. 29, 198–206 (1954).

The author proves a factor theorem for absolute Cesàro evaluability of integrals over infinite intervals. It is assumed that $\lambda>0$ and that all functions are real. In order that $k(t)$ be such that $\int_1^\infty x(t)k(t)dt$ is evaluable $|R,\lambda|$ whenever $\int_1^\infty x(t)dt$ is evaluable $|R,\lambda|$, it is necessary and sufficient that, for some constant $c\geqq 1$, (i) $k(t)$ be measurable and essentially bounded over $1\leqq t\leqq c$ and

$$(ii)\qquad k(t)=\frac{t}{\Gamma(\lambda)}\int_t^\infty(u-t)^{\lambda-1}h(u)du$$

almost everywhere over $t\geqq c$ where $u^{\lambda+1}h(u)$ is measurable and essentially bounded over $u\geqq c$. *R. P. Agnew.*

Watanabe, Yoshikatsu. Eine Verallgemeinerung des Abelschen Limitierungsverfahren bei Integralen. J. Gakugei, Tokushima Univ. (Nat. Sci.) 4, 21–29 (1954).

Emphasis being upon examples rather than upon formulation of theorems, it is supposed that

$$s(x)=\int_a^x f(u)du$$

and $g(u)$ is a real positive monotone function for which $g(u)\to\infty$ as $n\to\infty$ and

$$A(\lambda, g)=\int_a^\infty e^{-\lambda g(u)}f(u)du=\lambda\int_a^\infty e^{-\lambda g(u)}g(u)s(u)du.$$

The author studies the effect of different choices of $g(u)$ on the behavior of $A(\lambda, g)$ as $\lambda\to 0$ through real positive values and as $\lambda\to 0$ within sectors in the half plane where the real part of λ is positive. *R. P. Agnew* (Ithaca, N. Y.).

Watanabe, Yoshikatsu. Über die Verträglichkeit-Eigenschaft der gewissermassen erweiterten Cesàroschen und Abelschen Limitierungsverfahren bei Integralen. J. Gakugei, Tokushima Univ. (Nat. Sci.) 4, 30–38 (1954).

Keeping the notation of the preceding review, suppose that

$$C_k(x, g)=k\int_a^x\left(1-\frac{g(u)}{g(x)}\right)^{k-1}\frac{g'(u)}{g(x)}s(u)du$$
$$=\int_a^x\left(1-\frac{g(u)}{g(x)}\right)^k f(u)du.$$

A class of functions $g(u)$ is given such that $\lim_{x\to\infty} C_1(x, g)$ exists for each function in this class whenever it exists when $g(u)=u$. Another class is given such that the limit exists for each function in the class if and only if it exists when $g(u)=u$. It is shown that if $\lim_{x\to\infty} C_k(x, g)=L$, then $\lim_{\lambda\to 0} A(\lambda, g)=L$. *R. P. Agnew* (Ithaca, N. Y.).

van Wijngaarden, A. A transformation of formal series. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **56**=Indagationes Math. **15**, 522–533, 534–543 (1953).

A class of transformations of series is obtained as a generalization of the fact that the term-by-term Laplace transform of the Maclaurin series for $F(t)$ is an asymptotic series F for the transformed function $f(z)$. The generality depends upon (1) the use of integral transforms with more general kernels and (2) the introduction of an arbitrary "standard function" $S(t)$ whose expansion is used as a kind of comparison series. The transformed series TF has the form $\sum c_n s_n(z)$. The $s_n(z)$ are transforms of functions defined in terms of $S(t)$ and its derivatives, are independent of $F(t)$ and hence can be tabulated once for all after $S(t)$ has been chosen. Most of Part I is concerned with sufficient conditions on $S(t)$ and the kernel and on $F(t)$ in order for TF to converge to $f(z)$. A favorable choice of $S(t)$ may yield a series which is rapidly convergent in the entire half plane $\operatorname{Re} z>0$, in contrast to the Maclaurin series for $f(z)$ or the asymptotic series F whose usefulness is ordinarily restricted to small or large values of z respectively. Part II is concerned with the case of the Laplace kernel and the two particular choices for $S(t)$: $\exp(-t)$ and $1/(1+t)$. The former yields the Euler transformation of series and the latter is used to calculate values for certain special integrals such as the error integral. *P. W. Ketchum* (Urbana, Ill.).

Slipenčuk, K. M. On the generation of convergence of infinite products. Dopovidi Akad. Nauk Ukrain. RSR 1952, 110–114 (1952). (Ukrainian. Russian summary)

A real sequence $d_1, d_2, \cdots$ is such that the infinite product $\prod(1+d_k u_k)$ is convergent whenever the series $\sum u_k$ is convergent if and only if the two series $\sum(d_{k+1}-d_k)$ and $\sum d_k^2$ are both convergent. *R. P. Agnew* (Ithaca, N. Y.).

Azpeitia, A. G. Note on Dirichlet series. Revista Mat. Hisp.-Amer. (4) 13, 312–319 (1953). (Spanish)

Let $f(s)=\sum a_n e^{-\lambda_n s}$ be a general Dirichlet series. In the case $\lambda_n=n^\alpha$, $0<\alpha\leqq 1$, the author studies the problem of analytic extension of functions defined by such series under the assumption that the abscissa of convergence σ_c is finite. The methods employed are the same as those used by the reviewer [Duke Math. J. 14, 907–911 (1947); these Rev. **9**, 276] in the case $\lambda_n=\log n$. A typical theorem proved is the following. Let $g(re^{i\varphi})$ be holomorphic in and on the boundary of the region determined by the inequalities $-\pi/2\alpha<\psi_2<\varphi<\psi_1<\pi/2\alpha$, $\psi_1>0>\psi_2$, for a fixed $\alpha<1$ and there satisfy the condition $|g(re^{i\varphi})|<\exp[\delta r^\alpha]$ for some $\delta>0$ and all sufficiently large r. Then the function

$$f(s)=\sum g(n)\exp[-n^\alpha s],\quad s=\sigma+it,$$

is holomorphic in the region determined by the intersection of the half-planes $\sigma>\delta$, $\delta-\sigma\cos(\alpha\psi_1)+t\sin(\alpha\psi_1)<0$ and $\delta-\sigma\cos(\alpha\psi_2)+t\sin(\alpha\psi_2)<0$. The point at infinity is excluded. *V. F. Cowling* (Lexington, Ky.).

Newton, T. A. A note on a generalization of the Cauchy-Maclaurin integral test. Amer. Math. Monthly **61**, 331–334 (1954).

Sierpiński, W. Généralisation d'une formule de E. B. Escott pour les racines carrées. Bull. Soc. Roy. Sci. Liège **22**, 520–529 (1953).

Let $x>1$, let k_1 be the least positive integer for which $x>1+2/(k_1-1)$, and let x_1 be defined by

$$x=\left(1+\frac{2}{k_1-1}\right)x_1.$$

Applying these steps to x_1 and repeating the process gives

$$x=\left(1+\frac{2}{k_1-1}\right)\left(1+\frac{2}{k_2-1}\right)\cdots\left(1+\frac{2}{k_n-1}\right)x_n.$$

It is shown that $k_n\to\infty$ and $x_n\to 1$ and hence that x is represented as an infinite product. Numerous interesting inequalities, examples, and historical remarks are given. The integers k_n become infinite with exceptional rapidity; for example, if $x=3$ then $k_{13}>10^{4096}$. *R. P. Agnew.*

Fourier Series and Generalizations, Integral Transforms

Gosselin, R. P. On the theory of localization for double trigonometric series. Ann. Soc. Polon. Math. **24** (1951), no. 2, 49–77 (1954).

The author generalizes results of Z. Lepecki [Fac. Filos. Ci. Lit. do Paraná. Anuário 1940–1941, 159–187 (1942); these Rev. **4**, 157]. For series in one variable the theory was developed by Rajchman and Zygmund [Zygmund, Math. Z. 24, 47–104 (1925)] using formal multiplication of series, which is developed further in this paper for double series. One of the principal results is that if the series obtained by sufficient formal integration of the trigonometrical series

$$\sum_{m,n=-\infty}^{\infty} a_{mn}e^{i(mx+ny)}$$
$$(a_{m0}=a_{0n}=0,\ a_{mn}=o(|m|^\beta\cdot|n|^\sigma),\ \beta, \sigma>-1)$$

vanishes on a cross-shaped region $\{(x, y)|a\leqq x\leqq b$ or

$c \leqq y \leqq d\}$, then the series is uniformly summable (C, β, σ) in the Pringsheim sense to zero in every rectangle

$$\{(x, y) | a < a' \leqq x \leqq b' < b \quad \text{and} \quad c < c' \leqq x \leqq d' < d\}.$$

The methods are applicable to conjugate series and to series in more than two variables. *G. Klein.*

***Levitan, B. M. Počti-periodičeskie funkcii.** [**Almost periodic functions.**] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 396 pp. 10.75 rubles.

Have

In the preface of this book the author states that he does not intend to give a complete account of the theory of almost periodic functions, but that he has tried to include as much of the fundamental parts of the subject as the reader will need to be able to read the current literature 'freely'. With this object in mind the author follows the original papers rather closely, and he has not added any new theorems nor any essentially new methods to the theory. On the other hand, he has very skillfully selected the most convenient of the known proofs of the theorems. The book includes more different aspects of the theory than any earlier books although only fundamental matters are included. To compile all this material in a book of rather modest size, the author has had to make his exposition quite brief, and, owing to this, a few of the proofs are not easy to comprehend. The book includes a bibliography, which is also restricted to the fundamental parts of the theory.

Chapter 1 deals with ordinary almost periodic functions of a real variable. The main theorems are proved by the Weyl-Hammerstein method. Some theorems by Bochner [Proc. London Math. Soc. (2) **26**, 433–452 (1927)] on convergence of the Fourier series are included. The author also gives Bogolyubov's direct proof of the approximation theorem. Chapter 2 contains the theorems on the connection between the module of the function and the system of translation numbers. The mean motion and the distribution function are also discussed. Chapters 3 and 4 give a brief account of the N-almost periodic functions [Levitan, Uspehi Matem. Nauk (N.S.) **2**, no. 6(22), 174–214 (1947); these Rev. **10**, 293] and the application of these functions to the theory of systems of linear differential equations with almost periodic coefficients. Chapter 5 deals with the generalizations by Stepanov, Weyl and Besicovitch. The Riesz-Fischer theorem is proved but the characterization of B-almost periodic functions by their translation properties is omitted. Chapter 6 contains the theory of almost periodic functions on groups and the special results for Abelian groups and for topological groups. The problems concerning generalized translations are briefly discussed.

The last chapters of the book deal with analytic and harmonic almost periodic functions. A treatment of Jessen's generalization of Jensen's function is included. These last chapters are just copied from the original papers.

H. Tornehave (Copenhagen).

Kohlenberg, Arthur. Exact interpolation of band-limited functions. J. Appl. Phys. **24**, 1432–1436 (1953).

Let $F(w)$ be the Fourier transform of a function $f(t)$. If F vanishes except in the interval $|w| \leqq b$, then, as is well known, $f(t)$ is representable by the Lagrange interpolation formula relative to the function sin bt. A related interpolation formula is given for the case F vanishes except in the intervals $a \leqq |w| \leqq a+b$. The density of the interpolating points is the same in both cases. *R. J. Duffin.*

Hille, Einar. Some extremal properties of Laplace transforms. Math. Scand. **1**, 227–236 (1953).

A number of counter-examples of Laplace transforms with extremal properties are constructed. The following is shown. (a) There exists a Laplace transform $q(z)$ converging for Re $\{z\} > 0$, such that its Lindelöf mu-function satisfies $\mu(x; q) = 1-x$ for $0<x<1$, and $\mu(x; q) = 0$ for $x \geqq 1$. (b) There exists a Laplace transform $f(z)$, converging for Re $\{z\} > 0$, such that $\mu(x; f) \equiv 1$ for $x > 0$. (c) For each positive integer k there exists a function $f_k(z)$ represented by a Laplace integral summable (C, k) for $x > 0$, such that $\mu(x; f_k) \equiv k+1$ for $x > 0$. (d) There exists an entire function $f(z)$ with abcissa of (C, k) summability $\beta_k = 0$ for $0 \leqq k \leqq \infty$.

The main idea in the construction is first to construct a Laplace-Stieltjes transform with the desired properties and then to convert it into a Laplace transform multiplying by a slowly growing function like $[\log(z+2)]^{-1}$. In the first stage a Dirichlet series introduced by Bohr plays an essential role. Finally, it is noted in an addendum that some of the problems discussed were solved already by Jansson [Ark. Mat. Astr. Fys. **15**, no. 6 (1920)]. *S. Agmon.*

Riekstyn'š, È. Ya. Some new formulas for the Laplace transform. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 761–768 (1953). (Russian)

Using a general formula of operational calculus, the author discusses the inverse Laplace transforms of $(\alpha+p^{1/2})^{-n}$ and of $p^{-1/2}(\alpha+p^{1/2})^{-n}$. He gives many expansions, connections with the error function etc., and finally emerges with the known expressions of these inverse Laplace transforms in terms of parabolic cylinder functions of orders $-n-1$, $-n$, respectively. *A. Erdélyi* (Pasadena, Calif.).

Vasilache, Sergiu. Sur quelques formules fondamentales de la transformée de Laplace à deux variables. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3** (1951), 429–433 (1952). (Romanian. Russian and French summaries)

The author derives some of the elementary relations for the two-dimensional Laplace transforms. [See Voelker and Doetsch, Die Zweidimensionale Laplace-Transformation, Birkhäuser, Basel, 1950, Table A.1; these Rev. **12**, 699].

A. Erdélyi (Pasadena, Calif.).

Vasilache, Sergiu. Sur l'existence d'une solution de l'équation intégrale définie par la transformée de Laplace à deux variables indépendantes. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 209–218 (1951). (Romanian. Russian and French summaries)

"In this note the author proves that the integral equation of the first kind

$$\phi(p, q) = \int_0^\infty \int_0^\infty e^{-(px+qy)} f(x, y) dx dy \tag{1}$$

has a solution provided that ϕ is of the form

$$\phi(p, q) = \frac{1}{pq}\left\{f(0, 0) + \frac{h(p)}{p} + \frac{g(q)}{q} + \frac{H(p, q)}{pq}\right\}, \tag{2}$$

where h, g, H are analytic for Re $p>0$, Re $q>0$. The solution is unique, and is given by the complex inversion formula." (Paraphrase of the author's summary.) [Reviewer's remark: in this form the theorem is not true. The author seems to have established that the complex inversion formula holds provided ϕ is a double Laplace transform and is of the form (2).] *A. Erdélyi* (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Raljević, Šefkija. Sur une droite et sur un segment caractéristique dans les polygones des zéros des polynômes. Srpska Akad. Nauka. Zbornik Radova **35**. Mat. Inst. **3**, 89–94 (1953). (Serbo-Croatian. French summary)

Let the polynomial $P(z)$ have zeros z_j of multiplicity m_j $(j=1, 2, \cdots, p)$ and set

$$P_*(z)=P_n'(z)\Big/\prod_{\nu=1}^{p}(z-z_\nu)^{m_\nu-1}=a_n\prod_{j=1}^{\lambda}(z-z_j')^{m_j'}.$$

Then the points

$$\zeta_1=\sum_{j=1}^{p}m_jz_j\Big/\sum_{j=1}^{\lambda}m_j,\quad \zeta_2=\sum_{j=1}^{p}z_j/p,\quad \zeta_1^*=\sum_{j=1}^{\lambda}m_j'z_j'\Big/(p-1)$$

lie on a line and $p\zeta_2=(p-1)\zeta_1^*+\zeta_1$. In the special case $p=3$, a number of relationships are deduced which according to the author indicate that the segment joining ζ_1 and ζ_1^* stands as the generalized Euler segment for the triangle with vertices z_j and weights m_j $(j=1, 2, 3)$. The reviewer cannot agree since $\zeta_1=\zeta_1^*$ when $m_1=m_2=m_3$. Further, the author's assertion that $\zeta_1=\zeta_2$ only if $m_1=m_2=\cdots=m_p$ is erroneous as the example $P(z)=(z^2-1)^2z^2$ shows.

A. W. Goodman (Lexington, Ky.).

Tomić, Miodrag. Sur un théorème de L. Berwald. Srpska Akad. Nauka. Zbornik Radova **35**. Mat. Inst. **3**, 85–88 (1953). (Serbo-Croatian. French summary)

The author gives a simple and elegant proof of the following. Let $c_k>c_{k+1}>\cdots>c_n>0>c_0>c_1>\cdots>c_{k-1}$. Then the polynomial $f(x)=\sum_{\nu=0}^{n}c_\nu x^\nu$ has at most a simple zero on $|x|=1$ and k or $k-1$ zeros in $|x|<1$ according as $f(1)>0$ or $f(1)\leqq 0$. This generalization of Kakeya's theorem is contained in a theorem of Berwald [Math. Z. **37**, 61–76 (1933)] and in turn contains a theorem of A. Brauer [Math. Nachr. **4**, 250–257 (1951), Theorem 2; these Rev. **13**, 32].

A. W. Goodman (Lexington, Ky.).

Power, G. The associated Legendre polynomial. Math. Gaz. **38**, 115–116 (1954).

Sharpe, Charles B. A general Tchebycheff rational function. Proc. I. R. E. **42**, 454–457 (1954).

A Tchebycheff polynomial may be defined by a well-known trigonometric formula. It is shown that a simple modification of this formula gives rise to a rational function with prescribed complex poles. This function approximates zero in the interval $(-1, 1)$ with the Tchebycheff characteristic. Application is made to the design of low-pass RC filters having a specified ripple in the pass band. *R. J. Duffin.*

Alda, Václav. Completeness of polynomials for Poisson's distribution. Čehoslovack. Mat. Ž. **3**(78), 83–85 (1953). (Russian. English summary)

Let λ be a positive constant. If

$$\sum_{n=0}^{\infty}|a_n|^2\lambda^n/n!<\infty,\quad F(x^p)=\sum_{n=0}^{\infty}a_nn^p\lambda^n/n!=0$$

$$(p=0, 1, 2, \cdots),$$

then $a_n=0$ $(n=0, 1, 2, \cdots)$. Method of proof: Let

$$s_n(x)=(1-x)(2-x)\cdots(n-x)/n!.$$

It is shown in turn by elementary estimates that

$$0=F(s_n(x))=a_0+o(1),\quad 0=F(s_n(x-1))=a_1+o(1)\cdots.$$

W. H. J. Fuchs (Ithaca, N. Y.).

Motzkin, T. S., and Walsh, J. L. On the derivative of a polynomial and Chebyshev approximation. Proc. Amer. Math. Soc. **4**, 76–87 (1953).

Let E be a closed and bounded point set in the complex z-plane, $\mu(z)$ a positive function continuous on E. Let $f(z)$ be continuous on E. The Chebyshev polynomial of best approximation $t_m(z)$ is the polynomial of degree not exceeding m such that max $\{\mu(z)|f(z)-t_m(z)|\}$, for $z \in E$, is least. In particular, if $f(z)=z^{m+1}$, then $T_{m+1}(z)=z^{m+1}-t_m(z)$ is called the Chebyshev polynomial of degree $m+1$ for E with the weight function $\mu(z)$. This paper discusses the properties of these polynomials (explicit expressions, location of zeros) for the case that E is a finite set and contains just one point too many to preclude the case of interpolation when all deviations can be made to vanish. The following theorem is established: Let E be the finite set of distinct points $z_1, \cdots, z_n$ $(n>1)$, $\omega(z)=(z-z_1)\cdots(z-z_n)$. Let also the positive numbers $\lambda_1, \cdots, \lambda_n$ be given such that $\sum\lambda_i=1$. Then the Chebyshev polynomial $T_{n-1}(z)$ for E with the weight function $\mu(z_i)=(\lambda_i|\omega'(z_i)|)^{-1}$, is the polynomial

$$T_{n-1}(z)=\omega(z)\sum_{1}^{n}\frac{\lambda_i}{z-z_i}.$$

Since $T_{n-1}(z_i)=\lambda_i\omega'(z_i)$, it is observed that the arguments of the quantities $T_{n-1}(z_i)$ are independent of the choice of the λ_i and that the origin belongs to the convex hull of the $T_{n-1}(z_i)$. These results are applied to a discussion of the Chebyshev polynomial of best approximation $t_{n-2}(z)$ to an arbitrary function $f(z)$ and weight function $\mu(z)$ in the case that the set E has exactly n distinct points.

I. J. Schoenberg (Philadelphia, Pa.).

Natanson, I. P. On expansion of functions of two variables in series of orthogonal polynomials of a simple type. Doklady Akad. Nauk SSSR (N.S.) **91**, 1275–1277 (1953). (Russian)

The polynomials of the title are of the form

$$\omega_{n,m}(x, y)=\varphi_n(x)\psi_m(y)\quad (n, m=0, 1, 2, \cdots),$$

where the φ_n and ψ_m form orthonormal systems of polynomials with respect to weights $p(x)$ and $q(y)$, integrable and positive almost everywhere in the intervals $a\leqq x\leqq b$, $c\leqq y\leqq d$, respectively. Suppose $f(x, y)$ is integrable with respect to the weight $K(x, y)=p(x)q(y)$ on the rectangle $R=[a\leqq x\leqq b, c\leqq y\leqq d]$. The principal result is that f is expandable in the series

$$\sum_{i,k}c_{i,k}\omega_{i,k}(x, y),\quad c_{i,k}=\iint_R K(x, y)f(x, y)\omega_{i,k}(x, y)dxdy$$

at a point (x, y), where $\varphi_n(x)$ and $\psi_m(y)$ are uniformly bounded, provided

$$\iint_R K(u, v)\left[\frac{f(u, v)-f(u, y)-f(x, v)+f(x, y)}{(u-x)(v-y)}\right]^2$$

$$\times dudv<+\infty,$$

$$\int_a^b p(u)\left[\frac{f(u, y)-f(x, y)}{u-x}\right]^2du<+\infty,$$

$$\int_c^d q(v)\left[\frac{f(x, v)-f(x, y)}{v-y}\right]^2dv<+\infty.$$

G. Klein (So. Hadley, Mass.).

Special Functions

Have

Byrd, Paul F., and Friedman, Morris D. **Handbook of elliptic integrals for engineers and physicists.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Bd LXVII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. xiii+355 pp. DM 36.00; Ganzleinen DM 39.60.

The chief contents are: Definitions and fundamental relations; Reduction of algebraic, trigonometric and hyperbolic integrands to Jacobian elliptic functions (A); Tables of integrals of Jacobian elliptic functions (B); Elliptic integrals of the third kind; Certain related integrals; Derivatives; Expansions in series; Numerical tables. The last comprise tables of K, E, g with respect to the modular angle; K, K', E, E', g, g' with respect to k^2; $F(\phi, k)$; $E(\phi, k)$; $KZ(\beta, k)$, $\Lambda_0(\beta, k)$. The origin of these tables is not stated. The preface states that the book is for engineers and physicists. The main principle is that an integrand involving the square root of a cubic or quartic is to be looked for in (A) where it is reduced to an integrand involving Jacobian elliptic functions whose evaluation is then to be sought in (B). To complete the evaluation numerically, tables of the Jacobian functions are required, but it would not be gathered from this work that the only readily usable tables of these functions are the reviewer's "Jacobian elliptic function tables" [Dover, New York, 1950; these Rev. 13, 987]. As to notation it is rather astonishing to find the obsolete tn u used for Glaisher's sc u, especially as the great theoretical and mnemonic advantages of Glaisher's notation have been stressed by Neville [Jacobian elliptic functions, 2nd ed., Oxford, 1951; these Rev. 13, 24].

L. M. Milne-Thomson (Greenwich).

Humbert, P., et Agarwal, R. P. Sur la fonction de Mittag-Leffler et quelques-unes de ses généralisations. Bull. Sci. Math. (2) 77, 180–185 (1953).

This is a slightly expanded version of two notes [Humbert, C. R. Acad. Sci. Paris **236**, 1467–1468 (1953); these Rev. **14**, 872; Agarwal, ibid. **236**, 2031–2032 (1953); these Rev. **14**, 1084]. An added section gives some properties of the function

$$ {}_1E_{\alpha,\beta}=\sum_{m=0}^{\infty}\frac{x^m}{m!\Gamma(\alpha m+\beta)}. $$

A. Erdélyi (Pasadena, Calif.).

Vâlcovici, V. Sur la détermination des zéros des fonctions $J_{\pm\frac{1}{3}}(z)$, $J_{\pm\frac{2}{3}}(z)$ et de quelques autres fonctions qui s'y rattachent. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 3, 219–225 (1951). (Romanian. Russian and French summaries)

The author uses McMahon's formula [Watson, Bessel functions, Cambridge, 1944, sec. 15.53; these Rev. 6, 64] to compute the zeros of $J_{\pm\frac{1}{3}}(z)$, $J_{\pm\frac{2}{3}}(z)$, and adapts that formula for the computation of zeros of Airy integrals and of their derivatives. *A. Erdélyi* (Pasadena, Calif.).

Janković, Zlatko. On recurrence formulae for Bessel functions. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 8, 161–167 (1953). (Serbo-Croatian summary)

A discussion is given which is analogous to that presented by the author in a previous paper on Hermite's and Laguerre's equations [same Glasnik 8, 133–148 (1953); these Rev. 15, 423]. *N. D. Kazarinoff* (Lafayette, Ind.).

Ap Simon, H. G. A property of Bessel functions. Quart. J. Math., Oxford Ser. (2) 4, 282–283 (1953).

An expression of the form $I_n(at)K_n(bt)-K_n(at)I_n(bt)$ is expanded in a power series in t. The functions I_n and K_n are modified Bessel functions. *N. D. Kazarinoff.*

Rathie, C. B. Some infinite integrals involving Bessel functions. Proc. Nat. Inst. Sci. India 20, 62–69 (1954).

"The object of this paper is to evaluate a few infinite integrals involving modified Bessel functions of the second kind by the methods of Operational Calculus. Some of the results obtained are believed to be new and interesting". (Author's introduction.) The basis of the results is provided by the theorem that if

$$ f(x)\doteqdot\varphi(p) \quad\text{and}\quad p^{2-2\nu}e^{a/p}f(p)\doteqdot g(x), $$

then

$$ \varphi(p)=2pa^{\nu}\int_0^{\infty}(p+x)^{-\nu}K_{2\nu}\{2[a(p+x)]^{1/2}\}g(x)dx $$

[there are further hypotheses on the order of $g(x)$]. $\varphi(p)\doteqdot f(x)$ means that $\varphi(p)=p\int_0^{\infty}e^{-px}f(x)dx$.

N. D. Kazarinoff (Lafayette, Ind.).

Abramowitz, Milton. Regular and irregular Coulomb wave functions expressed in terms of Bessel-Clifford functions. J. Math. Physics 33, 111–116 (1954).

For tabular computations of the solutions of the differential equation

$$ \text{(1)}\qquad \frac{d^2y}{d\rho^2}+\left|1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right|y=0, $$

where L is a positive integer and η a continuous parameter, it is convenient to give series expansions in terms of Bessel functions. For $L=0$, and $t=2\eta\rho$ the differential equation transforms into

$$ \text{(2)}\qquad ty''+(\tfrac{1}{4}t\eta^{-2}-1)y=0 $$

with the solutions eventually in the forms

$$ \text{(3)}\quad \text{(regular)}\qquad \rho\Phi_0(\eta,\rho)=\lambda\sum b_nt^{n/2}I_n(2\sqrt{t}) $$

$$ \text{(4)}\quad \text{(irregular)}\qquad \Theta_0(\eta,\rho)=\lambda\sum(-1)^nb_nt^{n/2}K_n(2\sqrt{t}) $$

where $\lambda=1/2\eta$. The computations require therefore a single series of coefficients. Convergence is demonstrated for both (3) and (4) and extension is indicated for $L>0$.

E. Weber (Brooklyn, N. Y.).

Friedman, Bernard, and Russek, Joy. Addition theorems for spherical waves. Quart. Appl. Math. 12, 13–23 (1954).

Expansions or "addition theorems" for the spherical wave functions ... with reference to the origin O, have been obtained in terms of spherical wave functions with reference to the origin O'. (From the authors' summary.)

A. Erdélyi (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Allen, A. C., and Murdoch, B. H. A note on preharmonic functions. Proc. Amer. Math. Soc. 4, 842–852 (1953).

This paper concerns functions which are non-negative and preharmonic in the upper half-plane. A function defined at the integer points of the plane is termed preharmonic at a point if its value there is the mean of its values at the four

neighboring points. They prove a representation theorem analogous to that for positive harmonic functions [Loomis and Widder, Duke Math. J. 9, 643–645 (1942); these Rev. 4, 101]. Another theorem is the analog to the Phragmén-Lindelöf type theorem for positive harmonic functions [Loomis, Trans. Amer. Math. Soc. 53, 239–250 (1943); these Rev. 4, 199; A. C. Allen and E. Kerr, J. London Math. Soc. 28, 80–89 (1953); these Rev. 14, 469]. The main device in the proofs is an analog of the Poisson integral formula which essentially determines the function $f(m, n)$ in terms of its boundary values $f(m, 0)$ on the x-axis. In particular, it is seen thereby, that the series $\sum_{-\infty}^{\infty} f(m, 0)(1+m^2)^{-1}$ must converge. *R. J. Duffin* (Pittsburgh, Pa.).

Tsuji, Masatsugu. On the exceptional set of a certain harmonic function in a unit sphere. J. Math. Soc. Japan 5, 307–320 (1953).

Consider a harmonic function $u(P)=u(x, y, z)$ in the interior of the unit sphere S: $r=1$; suppose that

$$\iiint |\operatorname{grad} u(P)|^2 \frac{dv_P}{(1-r^2)^{1/2}} < \infty,$$

dv_P signifying the volume element. The author shows that there exists a set $E \subset S$ of Newtonian capacity zero, such that (1) for Q non-ϵ E, $\lim u(P)=u(Q)$ ($\neq \infty$) exists when P tends to Q from the inside of any Stolz domain, (2) for any line segment l which connects Q to a point of Δ, $\int_l |\operatorname{grad} u(P)|\,ds < \infty$. The starting point of the author's proof is his extension [Tôhoku Math. J. (2) 2, 113–125 (1950); these Rev. 12, 692] of Beurling's theorem on exceptional sets. *L. Sario* (Cambridge, Mass.).

Tsuji, Masatsugu. On criteria for the regularity of Dirichlet problem. J. Math. Soc. Japan 5, 321–344 (1953).

L'auteur passe en revue quelques résultats connus concernant les points-frontière irréguliers d'un domaine dont le complémentaire est compact (critère de Wiener et applications); le cas d'un domaine plan est étudié séparément. On énonce au début que si m_1 et m_2 sont des mesures positives dont les potentiels newtoniens U_1 et U_2 satisfont à $U_1 \leqq U_2$ quasi partout sur le support (compact) de m_1, alors $U_1 \leqq U_2$ partout: ce "principe du maximum" n'est pas vrai sans restriction sur m_1 (par exemple m_1 d'énergie finie, condition d'ailleurs réalisée dans les applications du texte). *J. Deny* (Strasbourg).

Komatu, Yûsaku, and Hong, Imsik. On mixed boundary value problems for a circle. Proc. Japan Acad. 29, 293–298 (1953).

The problem under discussion is the determination of a harmonic function in the unit disk, if its boundary values and the values of its normal derivative are given, respectively, on a subset α of the unit circumference and on its complement with respect to the latter. Explicit formulas are given for the case that α is either a single arc or consists of two disjoint arcs. Details of the calculations are to be published later. *Z. Nehari* (St. Louis, Mo.).

Komatu, Yûsaku. On mixed boundary value problems for functions analytic in a simply-connected domain. J. Math. Soc. Japan 5, 269–294 (1953).

The problem treated in the present paper is closely related to the mixed boundary-value problem discussed in the paper reviewed above, the only difference being that the values of the normal derivative are now replaced by the boundary values of the harmonic conjugate. Because of the Cauchy-Riemann equations, the two problems are equivalent if certain smoothness requirements are satisfied. It is to be noted, however, that while the problem involving the normal derivatives always has a solution, the problem of the present paper requires that the boundary data satisfy certain integrability conditions whose number is one less than that of the components of α. The author obtains explicit formulas for the solution in the case in which α consists of either one arc or two disjoint arcs. *Z. Nehari* (St. Louis, Mo.).

Šerman, D. I. On a connection of the fundamental problem of the theory of elasticity with a singular case of a problem of Poincaré. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 685–692 (1953). (Russian)

The determination of two harmonic functions φ and ψ in a finite multiply connected domain from the boundary conditions

$$\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}=f_1(s), \quad \frac{\partial\varphi}{\partial y}-\frac{\partial\psi}{\partial x}=f_2(s),$$

where the $f(s)$ are prescribed functions of the arc-parameter s on the boundary C of the domain, is reduced to the solution of a Fredholm integral equation. The boundary C consists of the exterior simple closed contour C_0 and several interior simple closed contours C_i not having common points, and the $f(s)$ are assumed to satisfy Hölder's condition on C. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Differential Equations

✱Bieberbach, Ludwig. Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage dargestellt. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd LXVI. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. ix +338 pp. DM 36.00; Ganzleinen DM 39.60. Have

This book is devoted to those parts of the theory of ordinary differential equations which can be treated with the help of function-theoretic methods and ideas. While much of the material can be found in the standard books on the subject (including the author's earlier volume on differential equations), and especially in the book of Ince [Ordinary differential equations, Longmans-Green, London, 1927], there is also much that has not found its way into other texts. The author does not aim at completeness and some topics, as, e.g., zero-free regions of solutions, are not treated at all. The book is written in the author's well-known lively (and sometimes snappy) style, and it makes good reading. Considerable care has been devoted to the simplification of known proofs and to the detailed discussion of phenomena which usually are given but casual attention.

The chapter headings are as follows. 1. The fundamental existence theorems. 2. Singular points of first-order equations. 3. The behavior of the solutions of

$$dw/dz=(aw+bz)(cw+dz)^{-1}$$

at the point (0, 0). 4. Non-essential singular points of the second kind. 5. First-order equations in the large. 6. Linear equations of the second order in the small. 7. Equations of

the Fuchsian type. 8. The hypergeometric equation. 9. The Bessel equation. 10. Equations of the Fuchsian type with four singular points. 11. Equations with periodic coefficients. 12. Selected topics in the theory of non-linear equations of the second order.

It is assumed that the reader is acquainted with the elements of the theory of functions of a complex variable. However, no previous knowledge of the theory of differential equations is required, and in those sections of the book which are mainly addressed to the beginner the author goes into considerable detail. *Z. Nehari* (Pittsburgh, Pa.).

Markus, L. Global structure of ordinary differential equations in the plane. Trans. Amer. Math. Soc. **76**, 127–148 (1954).

Let Σ denote a system of ordinary differential equations $\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$ defined in a simply-connected domain R of the Cartesian plane, it being assumed that $f(x, y)$ and $g(x, y)$ are of class C^1 and vanish simultaneously at isolated points only. These points, if there are such, are the only singular points of the family of solution curves of Σ, which elsewhere in R is locally homeomorphic to a family of parallel lines. Two systems Σ in R and Σ' in R' are equivalent in the sense of this paper if there is an orientation-preserving homeomorphism of R to R' carrying the curves of Σ onto the curves of Σ'. The objective is to enumerate the equivalence classes, find invariants determining the class, and to study special properties of certain of the classes.

An open set of R which is a union of solution curves will be called a parallel region if it is homeomorphic to (a) the lines y=constant in the plane, (b) concentric circles in the plane minus their center, or (c) the family of lines through a point in the plane minus the common point. If S is a solution curve then it is called a separatrix unless it is contained in a parallel region of R whose boundary consists of exactly two solution-curves which have common positive limit sets and common negative limit sets with each curve of this region. Let $\mathfrak{S}$ denote the collection of separatrices; then each domain of $R-\mathfrak{S}$ is called a canonical region. Finally a separatrix S is called a limit separatrix if it is in the closure of $\mathfrak{S}-S$. Among the principal results are the following. (1) Each canonical region is a parallel region. (2) Let $\mathfrak{S}'$ consist of the separatrices of Σ together with one curve from each canonical region and assume Σ_1, Σ_2 are systems in R_1 and R_2 with no limit separatrices other than critical points. Then Σ_1 and Σ_2 are equivalent if and only if $\mathfrak{S}_1'$ and $\mathfrak{S}_2'$ are homeomorphic under a map carrying $\mathfrak{S}_1$ onto $\mathfrak{S}_2$. (3) A stronger version of (2) based on the chordal systems of W. Kaplan [Duke Math. J. 7, 154–185 (1940); these Rev. 2, 322] is given for systems without critical points. (4) Let Σ have no limit separatrices and no parallel subregions of type (c). Then there exists a global integral for Σ, i.e., a function of class C^1 whose level curves are the solutions of Σ and which, moreover, is non-negative, vanishing exactly along the separatrices. *W. M. Boothby.*

On order

***Markus, L. A topological theory for ordinary differential equations in the plane.** Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 9, 8 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

This paper is a statement of the key results of the paper reviewed above, with sketches of the proofs.
W. Kaplan (Ann Arbor, Mich.).

Gagliardo, Emilio. Un'osservazione sui criteri di unicità per gli integrali di un'equazione differenziale ordinaria del primo ordine. Rend. Sem. Mat. Univ. Padova **23**, 214–223 (1954).

Let $f(x, y)$ be defined on a rectangle ($a\leqq x\leqq b$, $c\leqq y\leqq d$) so that $f(x, y)\leqq g(x)$ with $g(x)$ summable. A continuous function $y(x)$ on $a\leqq x\leqq b$ is said to be a solution of E: $y'=f(x, y)$ of type A if the right derivative D^+y exists and satisfies E everywhere, of type B if the right approximate derivative $D^+{}_{ap}y$ exists and satisfies E everywhere, of type C if the upper right derivative $\bar{D}^+y$ is finite and satisfies E everywhere, and of type D if the upper and lower right derivatives are everywhere both finite and $y'(x)$ (then existing almost everywhere) satisfies E almost everywhere. Sufficient conditions are given in the respective cases (generally similar and with similar weak Lipschitz conditions with respect to y) in order that a solution (if any) of the respective type be uniquely determined to the right of the initial point. *F. A. Ficken* (Knoxville, Tenn.).

Ważewski, T. Certaines propositions de caractère "épidermique" relatives aux inégalités différentielles. Ann. Soc. Polon. Math. **24** (1951), no. 1, 1–12 (1952).

Let $f(x, y)$ be a continuous function, and let $\tau(x)$ be the maximal integral of $y'=f(x, y)$, $y(a)=c$. An upper epidermis of τ consists of all points (x, y) such that $a<x<b$ and $\tau(x)<y<\tau(x)+\epsilon(x)$, where $\epsilon(x)$ is a function positive and continuous for $a\leqq x<b$. (The paper has $a<x<b$, but this obviously is not sufficient for the results.) Let E^ϵ be such an upper epidermis of τ, and assume that $\varphi(x)$ is continuous for $a\leqq x<b$, $\varphi(a)\leqq\tau(a)$, and $\varphi'(x)\leqq f(x, \varphi(x))$ whenever $(x, \varphi(x))\in E^\epsilon$; then $\varphi(x)\leqq\tau(x)$ for $a\leqq x<b$. This variant of a well-known result is most convenient in some applications because it is obvious that the appropriate φ would be differentiable at any point where its graph lay in an epidermis. *F. M. Stewart* (Providence, R. I.).

Łojasiewicz, S. Sur un théorème de Kneser. Ann. Soc. Polon. Math. **24** (1951), no. 2, 148–152 (1954).

Consider $\dot{x}=f(t, x, y)$, $\dot{y}=g(t, x, y)$ where f and g are continuous and bounded in E^3 (or appropriate assumptions for an open set in E^3). Let Z be the "zone of emission" (i.e., union of all solutions through p_0) from $p_0=(t_0, x_0, y_0)$. Let $Z\lambda$ be the trace of Z in the plane $t=\lambda$. Theorem: If there is a unique solution leading to the left ($t\to-\infty$) from each boundary point of Z in $t_0\leqq t\leqq\lambda$, then $Z\lambda$ is a continuum whose complement is simply connected. The author proves this extension of a theorem of Kneser and indicates the validity of the statement for higher dimensions. Moreover, given a plane continuum $Z\lambda$ which does not separate the plane, a (3-dimensional) differential equation is constructed for which this $Z\lambda$ is the required trace. *L. Markus.*

Szmydt, Z. Sur la structure de l'ensemble engendré par les intégrales tendant vers le point singulier du système d'équations différentielles. Bull. Acad. Polon. Sci. Cl. III. **1**, 223–227 (1953).

L'A. considera un sistema del tipo

$$dy_i/dt=F^{(i)}(y_1, \cdots, y_n) \quad (i=1, \cdots, n),$$

con le $F^{(i)}$ continue in un intorno del punto $\Theta=(0, \cdots, 0)$, nulle e derivabili in questo punto, con derivate quivi finite. L'A. indica una condizione sufficiente affinché le traiettorie di quel sistema, contenute in un intorno opportuno del punto Θ, topologicamente equivalente ad una sfera n-dimensionale

e chiusa, e tendenti a Θ al divergere di t verso $+\infty$, generino un insieme, omeomorfo ad una sfera chiusa m-dimensionale $(0<m<n)$ e dotato quasi ovunque di spazio lineare tangente. Secondo quella condizione, le radici caratterisitiche della matrice $\|F_{y_j}'(\Theta)\|$ hanno parte reale diversa da zero e lo spazio lineare caratteristico negativo di questa matrice ha m dimensioni e i numeri derivati parziali delle funzioni

$$F^{(i)}(y_1, \cdots, y_n)-\sum_{j=0}^{n} F_{y_j}^{(i)}(\Theta)y_j$$

sono tutti quanti limitati nell'intorno del punto Θ e continui quivi. *G. Scorza Dragoni* (Padova).

Szmydtówna, Z. Sur l'allure asymptotique des intégrales des équations différentielles ordinaires. Ann. Soc. Polon. Math. **24** (1951), no. 2, 17–34 (1954).

L'autore, utilizzando un metodo topologico di Ważewski, dimostra un teorema sul comportamento asintotico delle soluzioni di un sistema differenziale ordinario o non, teorema il quale comprende alcuni risultati classici di Perron, relativi ai sistemi lineari, e permette di estenderli e di precisarli, attraverso alcune sue conseguenze, nel senso che: il sistema

$$(1)\qquad y_i'=f_i(t)y_i+\sum_{j=1}^{n} g_{i,j}(t)y_j \quad (i=1, \cdots, n),$$

a coefficienti continui nella semiretta $T\leqq t<+\infty$, ammette $q+r$ soluzioni linearmente indipendenti verificanti la

$$\lim_{t\to+\infty}\frac{|y_1|^2+\cdots+|y_p|^2+|y_{p+q+1}|^2+\cdots+|y_{p+q+r}|^2}{|y_{p+1}|^2+\cdots+|y_{p+q}|^2}=0,$$

se, $\Re(f)$ indicando la parte reale di f,

$$\Re(f_1)\geqq\cdots\geqq\Re(f_p)>\Re(f_{p+1})\geqq\cdots\geqq\Re(f_{p+q})$$
$$>\Re(f_{p+q+1})\geqq\cdots\geqq\Re(f_{p+q+r}),$$

con $0<q<n$ e $p+q+r=n$, e se

$$\int_0^T\Re(f_p(t)-f_{p+1}(t))dt=\infty,\quad \int_0^T\Re(f_{p+q}(t)-f_{p+q+1}(t))dt=\infty,$$

$$\lim_{t\to+\infty} g_{i,j}/\Re(f_p-f_{p+1})=0,\quad \lim_{t\to+\infty} g_{i,j}/\Re(f_{p+q}-f_{p+q+1})=0;$$

il sistema (1) ammette n soluzioni linearmente indipendenti soddisfacenti alla

$$\lim_{t\to+\infty}\frac{|y_{k+1}|^2+\cdots+|y_n|^2}{|y_1|^2+\cdots+|y_k|^2}=0,$$

se $f_i(t)=\text{cost.}=\rho_i$, $\Re(\rho_1)>\Re(\rho_k)$ per $k=2, \cdots, n$, e se $\lim g_{i,j}(t)=c_{i,j}$, con $c_{i,1}=0$ per $k+1\leqq i\leqq n$, $1\leqq j\leqq k$ le parti reali delle radici carratteristiche della matrice $\|c_{i,j}\|$ $(i, j=1, \cdots, k)$ sono maggiori di quelle delle radici caratteristiche della matrice $\|c_{i,j}\|$ $(i, j=k+1, \cdots, n)$.

G. Scorza-Dragoni (Padova).

Morawetz, Cathleen S. Asymptotic solutions of the stability equations of a compressible fluid. J. Math. Physics **33**, 1–26 (1954).

The article is concerned with the asymptotic properties, for large λ, of certain systems of differential equations of the form

$$\frac{dz_j}{dt}=\sum_{p=1}^{6}\gamma_{jp}(x, \lambda)z_p, \quad j=1, \cdots, 6.$$

Of the functions $\gamma_{kp}(x, \lambda)$ it is assumed that $\lambda^{-2}\gamma_{kp}(x, \lambda)$ is regular analytic in both variables for $|x|\leqq x_0$, $|\lambda|\geqq\lambda_0$. It is well known [see, e.g., H. L. Turrittin, Contributions to the theory of nonlinear oscillations, vol. II, Princeton, 1952, pp. 81–116; these Rev. **14**, 377] that such differential equations possess certain formal solutions involving series in powers of λ^{-1}. The differential system studied by the author is of a special type, for its formal solutions are assumed to be of the form

$$e^{\lambda Q_p(x)}\sum_{r=0}^{\infty} f_{jpr}(x)\lambda^{-r}, \quad (p, j=1, \cdots, 6)$$

with

$$Q_1=-Q_2=\int_0^x\sqrt{b(t)}dt,\ Q_3=-Q_4=\int_0^x\sqrt{a(t)}dt,\ Q_5=Q_6=0,$$

where $a(x)$ and $b(x)$ have first order zeros at $x=0$. In addition, $|a'(0)|\neq|b'(0)|$.

Systems satisfying these conditions occur in the stability theory for compressible flows developed by L. Lees and C. C. Lin [NACA Tech. Note no. 1115 (1946); these Rev. **8**, 236]. The main purpose of the paper is to prove that the formal expansions used by Lees and Lin are asymptotic representations of actual solutions, and to supply more complete information about the "inner friction layers" mentioned by those authors. The occurrence of these layers is connected with the Stokes phenomena exhibited by the asymptotic representations of the solutions.

The method of the paper resembles that used by the reviewer for the analogous problem in the incompressible case [Ann. of Math. (2) **49**, 852–871 (1948); these Rev. **10**, 377], but the analysis is much more complicated. For instance, there are now twelve "Stokes lines" to be considered, instead of the three occurring in the simpler problem. The enumeration of the many solutions for which asymptotic expressions are derived and the description of the corresponding sectors of validity is too involved for a short review. It is an open question whether these sectors of validity are always the largest possible. However, for the solutions corresponding to $Q_p=0$ ($p=5$ or 6) a partial result in this direction is proved: If one of the functions $f_{jp0}(x)$ $(j=1, \cdots, 6)$ is multivalued around $x=0$, then a solution of the differential equation that tends to $f_{jp0}(x)$, as $\lambda\to\infty$, in a sector of the x-plane, diverges in a certain specified adjacent sector. *W. Wasow.*

Kimura, Toshifusa. Sur les points singuliers des équations différentielles ordinaires du premier ordre. Comment. Math. Univ. St. Paul. **2** (1953), 47–53 (1954).

The author extends his earlier study of the equation $dy/dx=R(x, y)$ [same Comment. **2**, 23–28 (1953); these Rev. **15**, 311] to the equation (1) $F(x, y, dy/dx)=0$, where F is a polynomial in y and dy/dx, with coefficients analytic in x. The singular points of the integrals of (1) are either movable or fixed. The fixed singular points ξ may be poles of order $\geqq1$ of a determination of dy/dx, whatever y, and they can be essentially singular points of the integrals; a fixed singular point may be transcendentally essential. It is proved that every integral $y(x)$ of (1) assumes all values, except for a finite number, in an arbitrary neighborhood of an essentially singular point ξ.

W. J. Trjitzinsky.

Sternberg, Robert L. A theorem on Hermitian solutions for related matrix differential and integral equations. Portugaliae Math. 12, 135–139 (1953).

The author presents a set of conditions under which a Hermitian matrix $W(x)$ satisfies the matrix differential equation

$$W'(x)+E(x, W(x))+F(x)=0, \quad a\leqq x<\infty,$$

if and only if $W(x)$ satisfies the integral equation

$$W(x)=\int_x^\infty E(t, W(t))dt+\int_x^\infty F(t)dt, \quad a\leqq x<\infty.$$

In particular, the result obtained generalizes a result of an earlier paper of the author [Duke Math. J. 19, 311–322 (1952); these Rev. 14, 50]. *W. T. Reid.*

Karanicoloff, Christo. Sur une équation différentielle d'ordre *n*. Acta Math. Acad. Sci. Hungar. 4, 237–242 (1953). (Russian summary)

The author considers the differential equation (*) $w^{(n)}(z)=f(z, z^p)w(z)$, where p is a positive constant and $f(z, u)$ is regular for $|z|\leqq R$, $|u|\leqq R^p$. He shows that (*) is solved by a series $w(z)=\sum_{\nu=0}^\infty c_\nu(z)z^{p\nu}$, converging in a deleted neighborhood of $z=0$, where the functions $c_\nu(z)$ can be determined by successively solving certain linear differential equations for which $z=0$ is a regular point. *Z. Nehari* (St. Louis, Mo.).

Hille, Einar. On the integration of Kolmogoroff's differential equations. Proc. Nat. Acad. Sci. U. S. A. 40, 20–25 (1954).

Given a matrix $A=(a_{ij})$ such that $a_{ij}\geqq 0$ $(i\neq j)$, $a_{ii}\leqq 0$ and $\sum_{j=1}^\infty a_{ij}=0$. The author discusses, by semi-group theory, the integration problem of Kolmogoroff's differential equations $p'_{ij}(t)=\sum_{k=1}^\infty a_{ik}p_{kj}(t)$ $(i, j=1, 2, \cdots)$. Let M be the Markoff algebra of matrices $B=(b_{ij})$ with the norm $\|B\|=\sup_i \sum_{j=1}^\infty |b_{ij}|<\infty$. Let $D(A)$ be the subspace of M for which the matrix product AB is defined as an element of M, and set $\overline{D(A)}=M_A$. The author considers the special case of triangular matrices $A=(a_{ij})$: $a_{ij}=0$ $(j>i)$. It is proved that the resolvent matrix $R(A)=R(\lambda, A)=(\lambda I-A)^{-1}$ exists, for $\lambda>0$, such that $\lambda R(\lambda, A)$ is a transition operator (non-negative matrix whose row sums are equal to one), though $D(A)$ is not dense in M. Hence there exists a semi-group of transition operators

$$P(t)B=\text{str}\lim_{n\to\infty}\exp(tnAR(n, A))B,$$

strongly continuous for $t\geqq 0$ when acting in M_A. The infinitesimal generator A_0 of $P(t)$ is the contraction of A to the subset of $D(A)$ in which $AB\in M_A$ so that

$$D(A^2)\subseteq D(A_0)\subseteq D(A).$$

$P(t)$ satisfies both equations of Kolmogoroff:

$$P'(t)=AP(t)=P(t)A.$$

While $P(t)B$ is the only solution of

$$Y'(t)=AY(t), \quad Y(t)\in D(A), \quad t>0, \quad \lim_{t\downarrow 0}\|Y(t)-B\|=0,$$

it is shown, by an example, that $P(t)$ is not the only solution of $Z'(t)=Z(t)A$, $\lim_{t\downarrow 0} z_{ij}(t)=\delta_{ij}$. In the course of the discussion, an explicit expression for the triangular $P(t)$ is obtained by making use of the inverse Laplace transform. *K. Yosida* (Osaka).

Pailloux, Henri. Sur la résolution des équations différentielles linéaires. C. R. Acad. Sci. Paris 238, 871–873 (1954).

The author sketches a theory of polynomial operators, polynomials in $p=d/dx$ whose coefficients are functions of x. The notions of division on the right, least common divisor on the right and greatest common multiple on the left of two or more polynomial operators are introduced. The main result is that the general linear system $\sum_{j=1}^n A_i^j u_j=\phi_i$, $i=1, 2, \cdots, n$, in which the A_i^j are polynomial operators, is equivalent to a system $Lu_1=\psi_1$, $H_k u_k=M_k u_1-\psi_k$, $k=2, 3, \cdots, n$ where L, H_k, M_k are certain polynomial operators, and the ψ_k are uniquely determined from derivatives of the ϕ_i. *C. E. Langenhop* (Ames, Iowa).

Böhm, Corrado. Nuovi criteri di esistenza di soluzioni periodiche di una nota equazione differenziale non lineare. Ann. Mat. Pura Appl. (4) 35, 343–353 (1953).

The differential equation considered is

$$\ddot{x}+\eta^{-1}\sin\theta\dot{x}+\sin x-\sin\theta=0,$$

where $\eta>0$ and θ are parameters. A solution is called stable if and only if $\lim_{t\to+\infty}\dot{x}=0$. It is proved that all solutions with $\eta\leqq\cos(\frac{1}{2}\theta)$ are stable and that all solutions with $\eta\geqq[\frac{1}{2}\pi(\sin\theta+\frac{1}{2}\pi\cos\theta)]^{1/2}$ are unstable. This improves considerably the inequalities derived by F. Tricomi [Ann. Scuola Norm. Super. Pisa (2) 2, 1–20 (1933)] and G. Seifert [Z. Angew. Math. Physik 3, 468–471 (1952); these Rev. 14, 647]. It is known [L. Amerio, Ann. Mat. Pura Appl. (4) 30, 75–90 (1949); these Rev. 11, 723] that the stability depends on certain properties of the integral curves through the singular point $x=\dot{x}=0$ in the $(x, \dot{x})$-plane. The author proves his criteria by applying a comparison theorem to these integral curves. *W. Wasow* (Los Angeles, Calif.).

Borůvka, O. Sur les intégrales oscilatoires des équations différentielles linéaires du second ordre. Čehoslovack. Mat. Ž. 3(78), 199–255 (1953). (Russian. French summary)

Let $Q(x)$ be continuous and negative for all real x and such that all the non-trivial solutions of the equation (a) $y''=Q(x)y$ are oscillatory, i.e. have infinitely many zeros with no finite limit point. Let $\cdots<\alpha_{-1}<\alpha_0<\alpha_1<\cdots$ be the ordered zeros of an integral $y_0(x)$ of (a). Since $y_0(x)$ is determined, except for a constant factor, by any one of its zeros, $\alpha_n(n=\pm1, \pm2, \cdots)$ is uniquely determined by α_0: $\alpha_n=\varphi_n(\alpha_0)$. The author calls $\varphi_n(x)$ the central dispersion (of the first kind) of index n; $\varphi_n(x)$ is monotone increasing and belongs to C_3. The $\varphi_n(x)$ form a cyclic group $\mathfrak{C}$ in the sense that $\varphi_n(\varphi_m(x))=\varphi_{n+m}(x)$, $\varphi_1(x)$ being the generator, and $\varphi_0(x)=x$ the unit element. The $\varphi_{2n}(x)$ $(n=0, \pm1, \cdots)$ form an invariant subgroup $\mathfrak{S}$ of $\mathfrak{C}$.

Let $U(x)$, $V(x)$ and $u(x)$, $v(x)$ be two fundamental systems of solutions of (a). Then a relation $\mathbf{p}$ is set up between the integrals $Y(x)=aU(x)+bV(x)$ and $y(x)=au(x)+bv(x)$ (a, b any real constants): $y=\mathbf{p}Y$. This leads to a relation $\alpha=\zeta(A)$ between the zeros A of Y and the zeros α (properly chosen) of y, which the author calls proper dispersion. A proper dispersion $\zeta(x)$ is in C_3 and either monotone increasing ("direct") or decreasing ("indirect"). The $\zeta(x)$ form a 3-dimensional group $\mathfrak{G}$ with $\zeta(x)=x$ as identity. The elements of $\mathfrak{G}$ are all the solutions of the equation of third order (b) $T''/T+\zeta'^2Q(\zeta)=Q(x)$, $T=|\zeta'|^{-1/2}$. The direct proper dispersions form an invariant subgroup $\mathfrak{P}$ of $\mathfrak{G}$ whose center is $\mathfrak{C}$. The group $\mathfrak{G}/\mathfrak{S}$ is isomorphic to the group of real unimodular matrices of order 2. Also considered are dis-

persions of the second, third and fourth kind referring to the zeros and the extrema of integrals of equation (a).
M. Golomb (Lafayette, Ind.).

Aymerich, G. Oscillazioni forzate di sistemi autosostenuti impulsivamente. Rend. Sem. Fac. Sci. Univ. Cagliari 22 (1952), 109–116 (1953).

The harmonic vibrations of the Rocard oscillator governed by the equations
$\dot{x}=-\epsilon f(x)+y,\ \dot{y}=-\omega^2x+A\sin\omega_1 t,\ f(x)=2\omega[x-x_0\cdot\operatorname{sgn}x]$,
where A and $\omega-\omega_1$ are of the order of the small parameter ϵ, are studied by the methods of Krylov-Bogolyubov. Stability of the solutions is also discussed. *J. L. Massera.*

Friedlander, F. G. On the oscillations of a bowed string. Proc. Cambridge Philos. Soc. 49, 516–530 (1953).
Keller, Joseph B. Bowing of violin strings. Comm. Pure Appl. Math. 6, 483–495 (1953).

These two papers report results, obtained simultaneously and independently, which are nearly identical and which also were obtained by using the same general method. The physical problem is formulated, as regards the violin string, in terms of the linear wave equation. The action of the bow is simulated by a force concentrated at a point of the string, and the force is assumed to depend on the relative velocity of string and bow. The law of force is that expected for solid friction, and is such that self-sustained oscillations are made possible when the parameters are in suitable ranges. The motion of the string is then studied by following progressing waves back and forth between the bowed point and fixed endpoints (in case there are such points). It turns out that quite a few interesting specific cases can be worked out more or less explicitly; e.g., an analysis can be made of parameter domains in which "noise" rather than a musical note occurs, conditions for a periodic motion can sometimes be derived, and questions of stability can be treated.
J. J. Stoker (New York, N. Y.).

★Klotter, K. Schwingungen mit endlich vielen Freiheitsgraden. Naturforschung und Medizin in Deutschland, 1939–1946, Band 4. Angewandte Mathematik, Teil II, pp. 67–85. Verlag Chemie, Weinheim, 1953. DM 10.00.

This is a terse but thorough survey of the German work in vibration systems having a finite number of degrees of freedom since the beginning of World War II. A bibliography of over two hundred references is appended. *E. Pinney.*

Höfinger, E. Zur Theorie der verallgemeinerten Pochhammerschen Differentialgleichung. Monatsh. Math. 57, 317–332 (1954).

G. D. Birkhoff [Proc. Amer. Acad. Arts Sci. 49, 519–568 (1913)] posed and virtually solved the so-called generalized Riemann problem: to construct a system of n linear differential equations, or a linear differential equation of order n, with prescribed singular points, prescribed determining factors and characteristic constants, and prescribed monodromic group. In general, accessory parameters will appear but in certain cases it turns out that the data are consistent and determine the differential equation (or system) uniquely. The author constructs an explicit example of this by discussing differential equations of the Pochhammer-Jordan type [Ince, Ordinary differential equations, Longmans-Green, London, 1927, sec. 18.4]. He also shows that in the case of differential equations of order two, his theory yields the hypergeometric equation, the confluent hypergeometric equation, and Weber's equation. *A. Erdélyi.*

★Lur'e, A. I. Nekotorye nelineĭnye zadači teorii avtomatičeskogo regulirovaniya. [Some nonlinear problems of the theory of automatic regulation.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 216 pp. 6 rubles.

Practically all the control systems considered in this monograph are described by equations of the form

$$\text{(a)}\qquad \dot{\eta}_k=\sum_{j=1}^n b_{kj}\eta_j+n_k\xi\qquad(k=1,\cdots,n)$$
$$\xi=f(\sigma),\quad \sigma=\sum_{j=1}^n p_j\eta_j-r\xi,$$

where the η_k are the controlled coordinates, ξ the coordinate of the regulating organ, $f(\sigma)$ the characteristic of the servomotor, b_{kj}, n_k, p_j, r constant parameters. The selection of problems and methods presented is motivated by the numerous original contributions of the author to this field. In Chapter 1 equations (a) are reduced to the canonical form

$$\text{(b)}\qquad \dot{y}_k=\lambda_k y_k+f(\sigma),\quad \dot{\sigma}=\sum_{j=1}^n\beta_j y_j-rf(\sigma)\quad(k=1,\cdots,n)$$

by a linear transformation $\eta\to y$. Here the λ_k are the zeros, assumed distinct, of $|b_{kj}-\lambda\delta_{kj}|$. The β_j are expressed explicitly in terms of the parameters of (a). In Chapter 2 sufficient conditions for the stability in the large of the solution $y_k=0$, $\sigma=0$ of equations (b) are established. They are derived from a properly chosen "Liapounoff function" $V(y,\sigma)$ which is positive for all values of y, $\sigma\neq0$ and whose time-derivative is negative for all functions $y(t)$, $\sigma(t)\neq0$ satisfying (b). Chapter 3 deals with the existence and calculation of the self-oscillations of system (b). Discussed are the Bogolyubov-Krylov method of harmonic balance, the Poincaré-Malkin method of expansion and casting-out of secular terms, and the author's solution in closed form for a few special functions $f(\sigma)$. In the last chapter the behavior of the solutions of (b) on the boundaries of the region of stability of the linear system $\dot{y}_k=\lambda_k y_k$ (either one of the λ's is 0 or there is a pair of pure imaginary λ's) is analyzed. Most of the results here are due to Bautin [Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 691–728 (1948); these Rev. 10, 456] and Aĭzerman [ibid. 14, 444–448 (1950); these Rev. 12, 181]. There are illustrative examples worked out in detail in all chapters. All the references given are to work in Russian. *M. Golomb* (Lafayette, Ind.).

Letov, A. M. Stability of control systems with two regulating organs. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 401–410 (1953). (Russian)

The equations describing the control system with two regulating organs are

$$\text{(a)}\qquad \dot{\eta}_k=\sum_{j=1}^n b_{kj}\eta_j+n_{k1}\xi_1+n_{k2}\xi_2\quad(k=1,\cdots,n),$$
$$\xi_i=f_i(\sigma_i),\quad \sigma_i=\sum_{j=1}^n p_{ij}\eta_j-r_{i1}\xi_1-r_{i2}\xi_2\quad(i=1,2),$$

where η_k are the controlled coordinates, ξ_i the coordinates of the regulating organs, $f_i(\sigma)$ given continuous functions describing the action of the servomotors, $f_i(0)=0$, $\sigma f_i(\sigma)>0$. If one assumes the zeros ρ_k of $|b_{\alpha\beta}+\delta_{\alpha\beta}\rho_\alpha|$ are distinct, a linear transformation $\eta\to x$ reduces (a) to the canonical form

$$\text{(b)}\qquad \dot{x}_k=-\rho_k x_k+u_{k1}f_1(\sigma_1)+u_{k2}f_2(\sigma_2),$$
$$\dot{\sigma}_i=\sum_{j=1}^n\beta_{ij}x_j-r_{i1}f_1(\sigma_1)-r_{i2}f_2(\sigma_2),$$

where the u_{ki}, β_{ij} are easily expressed in terms of the coefficients of (a).

System (a) is said to be absolutely stable if the solution $\eta_k=\xi_i=0$ is asymptotically stable no matter what the perturbations and the functions f_i are. The author establishes sufficient conditions for the absolute stability of (a), extending a criterion of Lur'e [see the book reviewed above] which applies to systems with one regulating organ. One set of sufficient conditions is: $\operatorname{Re}\rho_k>0, r_{11}>0, 4r_{11}r_{22}>(r_{12}+r_{21})^2$ and the system of equations quadratic in the a_k

$$\beta_{ik}+2a_k\sum_{j=1}^{n}\frac{a_j u_{ji}}{\rho_j+\rho_k}=0 \quad (k=1,\cdots,n;\ i=1,2)$$

has a solution with as many real and conjugate complex a_k as there are real and conjugate complex numbers among the ρ_k. The conditions are proved by exhibiting a "Liapounoff function" $V(x,\sigma)$ which is positive for all values of x, $\sigma\neq0$ and whose time-derivative is negative for all functions $x(t)$, $\sigma(t)\neq0$ satisfying (b). *M. Golomb.*

Weber, Heinrich E. Methodik der Berechnung von Regulierungen. Z. Angew. Math. Physik **4**, 233–260 (1953).

This is an exposition of well known procedures in the analysis of linear feedback control systems, in particular of the phase-attenuation method. Discussed are measures to increase stability, influence of disturbances, transient behavior and the effect of a finite time-delay within the system. *M. Golomb* (Lafayette, Ind.).

Brodin, Jean. Méthodes d'approximation en calcul fonctionnel. Ann. Télécommun. **9**, 1–8 (1954).

Suppose the output at time t of a control mechanism to the unit step input $u_\theta{}^t$ with step at time θ is given by $f_\theta{}^t$, where $f_\theta{}^t=0$ for $t<\theta$, is of bounded variation in θ uniformly with respect to t $(\theta\leqq t\leqq T)$, and as a function of t has only discontinuities of the first kind. Suppose also that the control mechanism is linear and continuous in an obvious way. Then the indicial function $f_\theta{}^t$ defines uniquely a linear operator $\bar{f}_\theta{}^t$ such that the output y^t to any input x^t which has only discontinuities of the first kind is given by $y^t=\bar{f}_\tau{}^t x^\tau$. The author develops an operational calculus for the operators $\bar{f}$ and, in particular, obtains the inverse of $\bar{u}+\bar{f}$ as $\bar{u}-\bar{f}+(\bar{f})^2-\cdots$. *M. Golomb* (Lafayette, Ind.).

Bellman, Richard, and Lehman, Shermann. On the continuous gold-mining equation. Proc. Nat. Acad. Sci. U. S. A. **40**, 115–119 (1954).

Let $\phi_1(t)$ satisfy $0\leqq\phi_1\leqq1$ for $t\geqq0$, define $\phi_2=1-\phi_1$, and let $x(t)$, $y(t)$, $p(t)$, $f(t)$ be defined by

$$x'=-\phi_1r_1x,\quad x(0)=x_0>0,\quad y'=-\phi_2r_2y,\quad y(0)=y_0>0,$$
$$p'=-p[\phi_1q_1+\phi_2q_2],\quad p(0)=1,$$
$$f'=p[\phi_1r_1x+\phi_2r_2y],\quad f(0)=0,$$

where r_1, r_2, q_1, q_2 are positive constants. The ϕ_1 which maximizes $f(\infty)$ satisfies

$$\phi_1=1,\quad \frac{r_2}{r_1+r_2},\quad 0$$

when $q_2r_1x>,\ =,\ <q_1r_2y$. The ϕ_1 which maximizes $f(T)$ for a fixed $T>0$ also assumes only these three values, and has at most two points of discontinuity within $(0,T)$. *D. Blackwell* (Washington, D. C.).

Rothman, M. The problem of an infinite plate under an inclined loading, with tables of the integrals of $\mathrm{Ai}(\pm x)$ **and** $\mathrm{Bi}(\pm x)$. Quart. J. Mech. Appl. Math. **7**, 1–7 (1954).

$\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ being two linearly independent solutions of the Airy equation $y''=xy$, the author shows that the problem of an infinite plate uniformly loaded at right angles to the infinite edges which are freely pivoted, the other two edges being free, depends on the solution of a differential equation of the form $d^3y/d\xi^3+A\xi dy/d\xi=B\xi+C$, and solves this equation in terms of integrals of $\mathrm{Ai}(x)$, $\mathrm{Bi}(x)$, and of Scorer's function $\mathrm{Gi}(x)$ [same J. **3**, 107–112 (1950); these Rev. **12**, 287]. 3 pages of numerical tables of $\int_0^x\mathrm{Ai}(\pm t)dt$, $\int_0^x\mathrm{Bi}(\pm t)dt$ are appended. *A. Erdélyi.*

Hamburger, Hans Ludwig. Remarks on self-adjoint differential operators. Proc. London Math. Soc. (3) **3**, 446–463 (1953).

A linear differential operator L, on one variable with a suitably restricted domain, determines a symmetric operator in Hilbert space if $\int_a^b(L(f)\bar{g}-f\overline{L(g)})dx=0$, a condition that does not require that the coefficients $p_\nu(x)$ in L be differentiable. This condition can also be expressed in terms of a fundamental set of solutions $u_1,\cdots,u_m$ of the differential equation $L(u)=0$. It is shown that this condition is equivalent to the existence for each such fundamental set of a skew symmetric constant matrix A such that $(Au)\cdot u^{(k)}=0$ for $k=0,\cdots,m-2$, $Au\cdot u^{(m-1)}=p_m(x)^{-1}$ where u denotes the vector $u_1,\cdots,u_m$, $u^{(k)}$ the kth derivative of this vector, and p_m the leading coefficient of L. The basis of the argument is the use of an inverse operator R for the operator L obtained from the "variation of parameters" procedure. This can be further explored in the case of operators with real coefficients; for instance one may obtain that if m is even, then the fundamental set of solutions may be chosen so that A has ones below the diagonal. The author also establishes the possible boundary conditions for the symmetric operators L which can be associated with a given differential operator. *F. J. Murray* (New York, N. Y.).

Bloh, A. Š. On the determination of a differential equation by its special matrix function. Doklady Akad. Nauk SSSR (N.S.) **92**, 209–212 (1953). (Russian)

The determination of $q(x)$ in $y''+(\lambda-q(x))y=0$, $x\geqq0$, from the spectral function $\rho(\lambda)$ associated with a boundary condition at $x=0$ has been considered by Gelfand and Levitan [Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 309–360 (1951); these Rev. **13**, 558]. Here the author considers the problem on $-\infty<x<\infty$ in which case $\rho(\lambda)$ must be replaced by a 2×2 matrix $T(\lambda)$. By use of the method of Gelfand and Levitan, q is determined from a knowledge of $T(\lambda)$. *N. Levinson* (Cambridge, Mass.).

Putnam, C. R. Note on a limit-point criterion. J. London Math. Soc. **29**, 126–128 (1954).

Let $L(x)=(px')'-fx$, where $p=p(t)>0$ and $f=f(t)$ denote real-valued, continuous functions on $0\leqq t<\infty$. Consider the solution $x(t,\lambda,\alpha)$ of $Lx+\lambda x=0$, λ real, which satisfies $x(0,\lambda,\alpha)=-\sin\alpha$ and $p(0)x'(0,\lambda,\alpha)=\cos\alpha$. Then L is in the limit-point case if and only if for some λ, x or $\partial x/\partial\lambda$ fails to be integrable squared on $[0,\infty]$. *N. Levinson* (Cambridge, Mass.).

Ziebur, A. D. On a double eigenvalue problem. Proc. Amer. Math. Soc. **5**, 201–202 (1954).

The following is proved: For the differential equation $(y'(p(x))'+\lambda q(x)(a+r(x))^{-1}p^{-2}(x)y=0$, $y(0)=y(c)=0$,

where $0 \leqq x \leqq l$, p, q, r are of class C'', and $p>0$, $q>0$, $r \geqq 0$, there exists for given $0<a_1<a_2$ a number a with $a_1 \leqq a \leqq a_2$ and a positive integer m such that m^2-1 is an eigenvalue belonging to a. The problem stems from the theory of rigidity of surfaces. *H. Busemann* (Copenhagen).

Levinson, Norman. The expansion theorem for singular self-adjoint linear differential operators. Ann. of Math. (2) **59**, 300–315 (1954).

Weyl-Stone's eigenfunction expansion, as completed by E. C. Titchmarsh [Eigenfunction expansions associated with second-order differential equations, Oxford, 1946; these Rev. **8**, 458] and K. Kodaira [Amer. J. Math. **71**, 921–945 (1949); these Rev. **11**, 438], for singular self-adjoint ordinary differential equations of the 2nd order has recently been given elementary proofs, independently by the author [Duke Math. J. **18**, 57–71 (1951); these Rev. **12**, 828], B. M. Levitan [Doklady Akad. Nauk SSSR (N.S.) **73**, 651–654 (1950); these Rev. **12**, 502] and the reviewer [Nagoya Math. J. **1**, 49–58 (1950); **6**, 187–188 (1953); these Rev. **13**, 39; **15**, 317]. Levitan's paper treats the equation of arbitrary order but it does not deal with the uniqueness of the expansion. K. Kodaira has extended his results to the equation of even order [Amer. J. Math. **72**, 502–544 (1950); these Rev. **12**, 103], and E. A. Coddington has treated, also by Hilbert space technique, the equation of arbitrary order [Proc. Nat. Acad. Sci. U. S. A. **38**, 732–737 (1952); these Rev. **14**, 278]. The present paper gives a self-contained elementary proof of the expansion theorem together with uniqueness. The method is similar to the author's previous paper, viz., he makes use of the Parseval relation of the eigenfunctions for the self-adjoint boundary-value problem pertaining to the bounded closed interval. *K. Yosida* (Osaka).

Conti, Roberto. I problemi ai limiti lineari per i sistemi di equazioni differenziali ordinarie: Teoremi di esistenza. Ann. Mat. Pura Appl. (4) **35**, 155–182 (1953).

The author establishes that there is at least one solution of the differential system

$$y_i'=F_i(x, y_1, \cdots, y_k)y_{i+1}+\phi_i(x, y_1, \cdots, y_k), \quad i=1, \cdots, k-1,$$
$$y_k'=\phi_k(x, y_1, \cdots, y_k),$$

for which the y_i assume prescribed values at ν_i points on $\alpha \leqq x \leqq \beta$, where $0 \leqq \nu_i \leqq k$ and $\nu_1+\cdots+\nu_k=k$, in case on C_∞: $\alpha \leqq x \leqq \beta$, $|y_j|<\infty$ $(j=1, \cdots, k)$, the F_j, ϕ_j are real-valued functions which satisfy the condition of Carathéodory, that is, they are continuous in $(y_1, \cdots, y_k)$ for fixed x, measurable in x for fixed $(y_1, \cdots, y_k)$, and in absolute value dominated by a Lebesgue integrable function on $\alpha \leqq x \leqq \beta$, while a certain associated determinant is bounded from zero on C_∞. This general existence theorem is shown to contain as special instances results for the problem of Nicoletti for equations and systems of equations of higher order, and results for certain types of boundary problems for equations and systems of equations involving parameters. The author also shows that the conclusion of his general existence theorem remains valid when the functions ϕ_j satisfy less stringent conditions than those mentioned above.
W. T. Reid (Evanston, Ill.).

Conti, Roberto. Problemi ai limiti lineari generali per i sistemi di equazioni differenziali ordinarie. Un teorema di esistenza. Boll. Un. Mat. Ital. (3) **8**, 153–158 (1953).

The author shows how the methods of the above paper may be used to prove the existence of at least one solution of the boundary problem

$$y_i'=\sum_{j=1}^{3} F_{ij}(x, y_1, y_2, y_3)y_j+\phi(x, y_1, y_2, y_3), \quad i=1, 2, 3,$$

$$\sum_{i=1}^{3} \gamma_{ji} y_i(\xi_j)=\eta_j, \quad j=1, 2, 3,$$

when on C_∞: $\alpha \leqq x \leqq \beta$, $|y_i|<\infty$ $(i=1, 2, 3)$ the functions F_{ij}, ϕ_i are real-valued and satisfy the condition of Carathéodory, $\alpha \leqq \xi_1 \leqq \xi_2 \leqq \xi_3 \leqq \beta$, the η_i, γ_{ji} are real constants, and a certain associated determinant is bounded from zero on C_∞. *W. T. Reid* [Evanston, Ill.).

Volpato, Mario. Sopra un problema di valori al contorno per l'equazione differenziale $y^{(n)}=f(x, y, y', \ldots, y^{(n-1)}, \lambda)$. Rend. Sem. Mat. Univ. Padova **23**, 224–244 (1954).

Sufficient conditions are given for the existence of at least one solution $(y(x), \lambda)$ of the problem

$$y^{(n)}=f(x, y, y', \cdots, y^{(n-1)}, \lambda),$$

$(a<x<b; \alpha<\lambda<\beta)$, $y(x_k)=C_k$ $(k=1, \cdots, n+1)$, where the C_k are real constants and $a \leqq x_1<x_2<\cdots<x_{n+1} \leqq b$. Equibounded equicontinuous approximations are obtained.
F. A. Ficken (Knoxville, Tenn.).

Volpato, Mario. Sugli elementi uniti di trasformazioni funzionali: un problema ai limiti per una classe di equazioni alle derivate parziali di tipo iperbolico. Ann. Univ. Ferrara. Sez. VII. (N.S.) **2**, 93–109 (1953).

Let Σ^* denote a Banach space of q-tuples of continuous functions on a rectangle. Let Φ denote a continuous mapping of Σ^* into a certain subspace Σ of Σ^*. The author obtains at least one fixed point of Φ by assuming that $\Phi(\Sigma^*)$ is bounded and (roughly) that the oscillation of the image functions is dominated by a nonhomogeneous linear expression in the oscillations of the original functions. Let $z_{r,s}$ denote $\partial^{r+s}z/\partial x^r \partial y^s$. The author considers a generalization of the characteristic initial value problem for the equation

$$z_{n,m}=f(x, y, z, \cdots, z_{\nu,\mu}, \cdots, z_{n,\mu}, \cdots, z_{\nu,m}, \cdots)$$
$$(\nu=0, \cdots, n-1; \mu=0, \cdots, m-1),$$

imposing a Lipschitz condition on f with respect only to those variables $z_{\nu,\mu}$ in which either $\nu=n$ or $\mu=m$. Two existence theorems in the large are presented, in one of which f is bounded while in the other this restriction is relaxed somewhat. Contact is established with similar work of Hartman and Wintner [Amer. J. Math. **74**, 834–864 (1952); these Rev. **14**, 475].

Correction: As the author points out on p. 228 of the paper reviewed above, the estimate (21) on p. 105 and the preceding continued estimate in the middle of p. 104 must each be augmented on the right, as (16) indicates, by $|G_{n,\mu}(x_1, y_1)-G_{n,\mu}(x_2, y_2)|$. This expression can be shown to tend uniformly to zero with the distance between (x_1, y_1) and (x_2, y_2), and thus can be absorbed into $H_{n,\mu}(\delta)$ in (23).
F. A. Ficken (Knoxville, Tenn.).

***Wilker, Peter. Über infinitesimale Berührungstransformationen zweiter Ordnung.** Dissertation, Universität Bern, 1950. Universitäts-Verlag Wagner, Innsbruck, 1952. 31 pp.

By an infinitesimal contact transformation of the second order (in the plane) is understood a transformation $\partial x/\partial t=X(x, y, p, q)$, $\partial y/\partial t=Y$, $\partial p/\partial t=P$, $\partial q/\partial t=Q$ which preserves the relations $dy-pdx=0$ and $dp-qdx=0$. From these conditions one obtains two linear equations in dx and

dq whose coefficients are linear functions of the partial derivatives of X, Y, P, Q. If the rank of this system is 2 the result is trivial, while if the rank is 0, the transformation is an extended contact transformation of order 1. The case of rank 1 leads to a transformation which may be reduced to $x'=1$, $y'=p$, $p'=q$, $q'=F(x, y, p, q)$ from which the author constructs two linear differential operators, $\mathfrak{A}f$ and $\mathfrak{B}f$, and shows that the commutativity of these operators is a necessary condition on X, Y, P, Q. Sufficiency is proved by actual construction. The author also generalizes the theorem on ordinary differential equations of the first order to those of order 2, showing that if two independent contact transformations of the second order leave an ordinary differential equation of the second order invariant, the equation may be solved by two quadratures. *M. S. Knebelman.*

Henrici, Peter. Zur Funktionentheorie der Wellengleichung. Mit Anwendungen auf spezielle Reihen und Integrale mit Besselschen, Whittakerschen und Mathieuschen Funktionen. Comment. Math. Helv. 27 (1953), 235–293 (1954).

In the first part of this paper the author studies the partial differential equation

$$(1) \qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2v}{y}\frac{\partial u}{\partial y}+k^2u=0$$

in terms of the variables $z=x+iy$, $z^*=x-iy$. He first proves that given any analytic function $g(z)$, (1) has a unique analytic solution $U(z, z^*)$ which is symmetric in z and z^* and reduces to $g(z)$ when $z=z^*$. $U(z, z^*)=\theta_v[g; z, z^*]$ defines an integral operator which is given explicitly. It is shown that every regular solution of (1) can be extended to an analytic function of the complex variables z, z^*, and also that given any analytic function $f(z)$, (1) has a unique analytic solution $U(z, z^*)$ which reduces to $f(z)$ when $z^*=0$. $U=\Omega_v f$ defines the integral operator, Ω_v.

In the second part, the operators θ_v and Ω_v are used to obtain theorems analogous to the fundamental theorems of complex variable theory. $\Omega_v z^m$ and $\theta_v z^m$, $m=0, 1, 2, \cdots$, form a base for the analytic solutions of (1). The elements of this base are exhibited as the well-known products of Bessel functions and Gegenbauer polynomials obtained by the separation of variables in polar coordinates.

In the third part the author considers several solutions of (1) in various systems of curvilinear coordinates. According to the results of the second part, each of these solutions can be expanded in a series of products of Bessel functions and Gegenbauer polynomials. The author gives numerous expansions of this kind, some of them known while others are generalizations of known expansions. *A. Erdélyi.*

Kapilevič, M. B. On fundamental solutions of an equation of hyperbolic type. Doklady Akad. Nauk SSSR (N.S.) 91, 719–722 (1953). (Russian)

For the domain D $(y>x>0)$ the author discusses the equation

$$(*) \qquad z_{xy}-f(y-x)(z_y-z_x)=0, \quad f(y-x)=-\frac{1}{2\sqrt{K}}\frac{\partial\sqrt{K}}{\partial x}$$

where

$$\sqrt{K}(y-x)=b_1(y-x)^{1/3}+b_3(y-x)^{3/3}+b_5(y-x)^{5/3}+\cdots.$$

The domain D is divided into six parts by starting with two points on the line $y=x$ and drawing the characteristics of (*) through them. For such a domain the Riemann function for the Cauchy problem with data on the singular line is known. The author calls the Hadamard function the fundamental solution which solves a mixed boundary-initial value problem. Integral representations for the fundamental solution are tabulated for various combinations of the six domains. The author also considers certain representations for the equation

$$z_{xy}-\frac{1}{6(y-x)}(z_y-z_x)-\frac{b^2}{4}z=0.$$

M. H. Protter (Berkeley, Calif.).

Stellmacher, Karl L. Ein Beispiel einer Huyghensschen Differentialgleichung. Nachr. Akad. Wiss. Göttingen. Math. Phys. Kl. Math.-Phys. Chem. Abt. 1953, 133–138 (1953).

A linear hyperbolic differential equation of the second order is said to be "Huyghensian" if its solution of the Cauchy initial-value problem depends, at the point P, not on all the Cauchy data, but only on that part which is defined on the intersection of the characteristic conoid of vertex P with the data support manifold. The author shows the existence of Huygensian equations of the form,

$$\sum_{i=1}^{5}\partial^2u/\partial x_i^2-\partial^2u/\partial x_6^2=c(x_1, \cdots, x_6)u,$$

in which $c\neq0$. *D. C. Lewis* (Baltimore, Md.).

Garnir, H. G. Sur la propagation de l'onde émise par un point dans un angle ou un dièdre parfaitement réfléchissant et le problème analogue pour la conduction de la chaleur. Bull. Soc. Roy. Sci. Liège 21, 328–344 (1952).

In a previous paper [same Bull. 21, 119–140, 207–231 (1952); these Rev. 15, 130] the author has derived the Green's functions for the operators $\Delta-p^2/c^2$ and $\Delta-p/k$. Now, by using the inverse Laplace transform, he obtains the Green's functions for the operators $\Delta-c^{-2}\,\partial^2/\partial t^2$ and $\Delta-k^{-1}\,\partial/\partial t$ in a dihedral angle. The results are expressed as the sum of the field produced by a source in infinite space and of the field produced by the planes bounding the angle. *B. Friedman* (New York, N. Y.).

Owens, O. G. A boundary-value problem for analytic solutions of an ultrahyperbolic equation. Duke Math. J. 21, 29–38 (1954).

The author considers the equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=\frac{\partial^2 u}{\partial y_1^2}+\frac{\partial^2 u}{\partial y_2^2}$$

over the topological region $C(x)\times C(y)$, where x denotes the point (x_1, x_2) and $C(x)$ denotes the region $x_1^2+x_2^2\leq1$. A solution is said to be an analytic admissible function if $\Delta_x^n\Delta_y^m u(x, y)$ and all its first derivatives are continuous in (x, y) on $C(x)\times C(y)$, $m, n=1, 2, \cdots$ and

$$|\Delta_x^n\Delta_y^m u(x, y)|\leq4^{n+m}\tau(n, m)$$

with $\tau(n, m)$ tending to zero for n and m tending to infinity. Here Δ_x^n, Δ_y^m are the two-dimensional iterated Laplace operators. An analytic admissible solution is proved to exist and to be uniquely determined by its boundary values $g(x^*, y)$ for $|x^*|=1$, where $g(x^*, y)$ is assumed to satisfy conditions analogous to those above. The solution is obtained in the form of an infinite series by a method of successive approximations. *Maria Steinberg.*

Garabedian, P. R., and Shiffman, Max. On solution of partial differential equations by the Hahn-Banach theorem. Trans. Amer. Math. Soc. 76, 288–299 (1954).

The object of the paper is to construct Green's and Neumann's function for an equation of the form

$$(1)\qquad M(u)=\Delta u-Pu=0$$

by the use of the Hahn-Banach extension theorem. We describe the procedure in the case of the Neumann function: Let D be a domain of the $z=x+iy$-plane whose boundary consists of a finite number of simple closed analytic curves C. Let B be the space of all functions $f(z)$ which are continuous in $D+C$. With the norm $\|f\|$ defined as $\max|f(z)|$, B is a Banach space. The function $P=P(z)$ in (1) is supposed to be continuously differentiable. Let B_1 be the subspace of those functions u of B which have first and second continuous derivatives in $D+C$ and for which $\partial u/\partial\nu=0$ on C where ν denotes the inner normal to C. The image of B_1 under the operator $M(u)$ of (1) is denoted by B'. Let (speaking formally) $u=L(v)$ denote the inverse of the operator M and $u(w)=L_w(v)$ the value of the inverse at the point w of D such that (2) $u(w)=L_wM(u)$. It is proved that (2) actually defines uniquely a bounded linear functional $L_w(f)$ for $f\subset B'$. By the Hahn-Banach theorem the linear functional $L_w(f)$ is then extended (with the same bound) to all of B. To obtain Neumann's function, the authors construct now in a definite way a function $S(t,z)$ (parametrix) which as function of t has a logarithmic singularity at $t=z$, for which, however, $f=MS$ is continuous even at $t=z$. The extended functional L_w is then well defined at $f=M_1S(t,z)$ where the index 1 indicates that $M=M_1$ operates on the first variable of S. It is then shown that the function $N(w,z)$ defined by $N(w,z)=S(w,z)-L_wM_1S(t,z)$ is Neumann's function.

The treatment of Green's function is analogous. It is carried through under less restrictive conditions on the boundary and for three independent variables. Various remarks are made concerning possible generalization of the method (e.g., regarding the form of the differential equation, the number of independent variables, or to Riemannian manifolds).
E. H. Rothe (Ann Arbor, Mich.).

Müller, Claus. The behavior of the solutions of

$$\Delta U=F(x,U)$$

in the neighborhood of a point. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. BR-5, i+18 pp. (1954).

The purpose of this paper is to investigate the local behavior of the solutions of (1) $\Delta U=F(x,U)$ and in particular the behavior of the solutions of (2) $\Delta U+k^2(x)U=0$, where x denotes the spatial vector $(x_1,x_2,\cdots,x_p)$ in Euclidean spaces of p dimensions and Δ is the Laplacian operator in that space. The main result of the paper is as follows. Let U have continuous second derivatives in $|x|\leqq\alpha$. Let U satisfy the inequality

$$(3)\qquad \int_{\Omega_R}|\Delta U|^2ds\leqq c\int_{\Omega_R}|U|^2ds$$

uniformly in R for $0<R\leqq\alpha$, where Ω_R denotes the hypersphere $|x|=R$. Then if

$$\int_{\Omega_R}|U|^2ds=o(R^n)$$

for all n as $R\to0$, then U vanishes identically.

The conditions of this theorem are evidently satisfied if U is a solution of (2) where $k(x)$ is uniformly bounded; and so any solution of (2) which vanishes more strongly than any power of the distance vanishes identically, a result previously proved only with the assumption that $k(x)$ is analytic in x.

More generally, it follows that if U_1 and U_2 are solutions of (1) which have continuous derivatives in $|x-x_0|\leqq\alpha$, if $U_1(x_0)=U_2(x_0)$, if $F(x,T)$ satisfies

$$|F(x,T_1)-F(x,T_2)|\leqq L|T_1-T_2|$$

in a neighbourhood of x_0 and $U_1(x_0)$, then if

$$\int_{\Omega_R}|U_1(x)-U_2(x)|^2ds=o(R^n),$$

where Ω_R is $|x-x_0|=R$, holds for all n as $R\to0$, then U_1 is identically equal to U_2.
E. T. Copson.

Vvedenskaya, N. D. On a boundary problem for equations of elliptic type degenerating on the boundary of a region. Doklady Akad. Nauk SSSR (N.S.) 91, 711–714 (1953). (Russian)

Equations of the form

$$y^m\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+a(x,y)u_y+b(x,y)u_x+c(x,y)u=0,$$

$$y^m\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial x^2}+a(x,y)u_y+b(x,y)u_x+c(x,y)u=0,$$

are considered in a domain D in the half-plane $y>0$. The coefficients are assumed analytic with $c\leqq0$, $m>0$. The closed curves forming the boundary of D are assumed to consist of segments of the x-axis and certain arcs Γ in the upper half-plane ending on these segments. For such a domain the boundary conditions

$$\frac{\partial u}{\partial\nu}+A(x,y)u=\varphi(x,y)\text{ on }\Gamma,$$

$$u(x,0)=f(x),$$

where ν is a direction making an acute angle with the interior normal to Γ, $A\leqq0$, and $\max(A(x,y)+c(x,y))<0$ at the end-points of the segments on the x-axis. Under the above conditions, together with certain smoothness hypotheses, existence and uniqueness theorems are established.
M. H. Protter (Berkeley, Calif.).

Košelev, A. I. Newton's method and generalized solutions of nonlinear equations of elliptic type. Doklady Akad. Nauk SSSR (N.S.) 91, 1263–1266 (1953). (Russian)

For an elliptic equation of the form

$$(*)\qquad \sum_{\substack{i+k=2\\ i,k\geqq0}}A_{ik}(x_j;p_j,u)\frac{\partial^2u}{\partial x_1{}^i\partial x_2{}^k}+B(x_j;p_j,u)=0,\quad j=1,2,$$

$p_j=\partial u/\partial x_j$, defined in a domain Ω the author discusses the homogeneous boundary-value problem. A lemma is stated concerning the possibility of reducing (*) to a functional equation and thus apply a generalization of Newton's method for obtaining a solution. For linear equations and the minimal surface equation estimates are obtained for the maximum of the solution, the first derivatives, and the L_p norms of the second derivatives.
M. H. Protter.

Pini, Bruno. Sulle equazioni lineari a derivate parziali d'ordine $2n$ di tipo ellittico e sui sistemi ellittici di equazioni lineari del secondo ordine sopra una superficie chiusa. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 176–195 (1952).

This paper establishes for linear elliptic equations $\mathfrak{L}u=f$ of order $2n$ ($n\geqq 1$) on a closed surface S results obtained for $n=1$ by G. Cimmino [Ann. Scuola Norm. Super. Pisa (2) **7**, 73–96 (1938)]. Let $\mathfrak{M}$ be the adjoint of $\mathfrak{L}$ and let D be any sufficiently small neighborhood on S of a point Q of S. Using methods due to E. E. Levi, the author constructs on D a function V which vanishes, along with its first $2n$ derivatives, on the boundary of D, and is such that $\mathfrak{L}V$ has a logarithmic singularity at Q. It is shown, under suitable conditions, that if $\mathfrak{M}v=f$ then $v(Q)=c\iint_D(v\mathfrak{L}V-fV)dP$ (c a constant depending on $\mathfrak{L}$ and Q), and if this is true for all sufficiently small D and corresponding V then $\mathfrak{M}v=f$. It is also shown, essentially, that if $w \in L^p$ and w is orthogonal to $\mathfrak{L}u$ for every $u \in C^{2n}$ then $\mathfrak{M}w=0$. An "alternative" is indicated, and an extension to systems is given.

F. A. Ficken (Knoxville, Tenn.).

Èĭdel'man, S. D. Estimates of solutions of parabolic systems and some of their applications. Mat. Sbornik N.S. **33**(75), 359–382 (1953). (Russian)

The author considers parabolic systems of the form

$$(*)\qquad \frac{\partial^{n_i}u_i}{\partial t^{n_i}}=\sum_{j=1}^{N}\sum_{(k_s)}A_{ij}{}^{(k_0,k_1,\cdots,k_n)}(t)\frac{\partial^{k_0+k_1+\cdots+k_n}u_j}{\partial t^{k_0}\partial x_1{}^{k_1}\cdots\partial x_n{}^{k_n}},\qquad i=1,2,\cdots,N,$$

where the coefficients are continuous complex-valued functions of the real variable t. Under certain hypotheses (too complicated to state) on the growth of the quantities

$$\max_{t_0\leqq t\leqq T}\left|\frac{\partial^k u_i(x,t)}{\partial t^k}\right|,\quad i=1,\cdots,N;\ k=1,\cdots,n_i-1,$$

together with some smoothness conditions a uniqueness theorem for the Cauchy problem for the system (*) is established. The existence of the solution is also established but under somewhat stronger hypotheses. The method consists in getting strong estimates for the Green's matrix for the associated system of ordinary differential equations. These techniques are based on generalizations of the work of Petrowsky [Bull. Univ. d'Etat Moscou. Ser. Internat. Sect. A. Math. Méc. **1**, no. 7, 1–74 (1938)]. Applications are given to the case of systems with constant coefficients and to certain classical problems for a single parabolic equation.

M. H. Protter (Berkeley, Calif.).

Prodi, Giovanni. Problemi al contorno non lineari per equazioni di tipo parabolico non lineari in due variabili—soluzioni periodiche. Rend. Sem. Mat. Univ. Padova **23**, 25–85 (1954).

Let R denote the set of points $0<x<l$, $0<t\leqq T$. Let $X(x)$, $\varphi_1(t,u)$, $\varphi_2(t,u)$ be given continuous functions for $0\leqq x\leqq l$, $0<t\leqq T$, $-\infty<u<\infty$, and such that $|\varphi_i(t,u)|\leqq H(|u|)t^{-\beta}$, $i=1,2$. Here H is a non-decreasing function and $0\leqq\beta<\frac{1}{2}$. Further, let $F(x,t,u,u_x)$ be a continuous function for (x,t) in R, $-\infty<u$, $u_x<+\infty$, which at each point of its definition satisfies a Hölder condition and also the condition $|F(x,t,u,u_x)|\leqq H(|u|)\{t^{-\alpha}+|u_x|^{2\alpha}\}$, $0\leqq\alpha<1$. For T sufficiently small the following non-linear boundary value problem is shown to have at least one solution $u=u(x,t)$

$$u_t=u_{xx}+F(x,t,u,u_x)\quad(\text{in } R),$$
$$u(x,0)=X(x)\quad(0\leqq x\leqq l),$$
$$u_x(0,t)=-\varphi_1(t,u(0,t)),\quad u_x(l,t)=\varphi_2(t,u(l,t))\quad(0<t\leqq T).$$

Under certain additional conditions on the functions F, φ_1 and φ_2 the restriction T sufficiently small can be removed.

Conditions, too many to state here, are given that will insure the boundary-value problem $u_t=u_{xx}+f(x,t,u,u_x)$; $u(0,t)=-g_1(t,u(0,t))$, $u(l,t)=g_2(t,u(l,t))$, $-\infty<t<\infty$, to have at least one solution of period T assuming f, g_1, g_2 have period T in the variable t.

F. G. Dressel.

Dobryšman, E. M., and Belousov, S. L. On the two-layer problem of heat conduction, air-earth. Doklady Akad. Nauk SSSR (N.S.) **93**, 1011–1014 (1953). (Russian)

The problem of heat transfer between air and earth is considered with the stipulation that the air mass varies slowly. This leads to the system of differential equations

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial Z}\left[k_0\left(\frac{Z}{Z_0}\right)^{1-1/\nu}\frac{\partial T}{\partial Z}\right]\quad(Z>0),$$
$$\frac{\partial T}{\partial t}=k^*\frac{\partial^2 T}{\partial Z^2}\quad(Z<0),$$

subject to the conditions

$$T(Z,0)=\Phi(Z),\quad T^*(Z,0)=\Phi^*(Z),\quad \text{at } t=0,$$
$$T(0+,t)=T^*(0-,t),$$
$$\lim_{Z\to 0}\left[-\lambda_0\left(\frac{Z}{Z_0}\right)^{1-1/\nu}\frac{\partial T}{\partial Z}+\lambda^*\frac{\partial T^*}{\partial Z}+\mu T\right]=F(t),$$

$T(\infty,t)$ and $T(-\infty,t)$ are bounded. Here T and T^* represent the variations of temperatures from some average values $\bar{T}$ and $\bar{T}^*$ in air and earth, respectively; and Z is positive in the upward direction. λ^*, k^* are the coefficients of heat conduction and diffusion in earth, λ_0, k_0 the same for air at the level Z_0, and ν, μ are parameters characterizing the turbulent interchange of heat in air and radiation of heat from the surface of the earth. A change of variables is made through the substitution

$$T=\frac{Q}{\mu}\theta(\tau,\zeta),\quad T^*=\frac{Q}{\mu}\theta^*(\tau,\xi),\quad \Phi=\frac{Q}{\mu}\varphi(\zeta),\quad \Phi^*=\frac{Q}{\mu}\varphi^*(\xi),$$
$$\frac{Z}{Z_0}=\left(\frac{\lambda_0}{\mu Z_0}\right)^{\nu}\zeta,\ Z>0,\quad \frac{Z}{Z_0}=-\left(\frac{\lambda_0}{\mu Z_0}\right)^{(1+\nu)/2}\left(\frac{k^*}{k_0}\right)^{1/2}\xi,\ Z<0,$$
$$t=\left(\frac{\lambda_0}{\mu Z_0}\right)^{1+\nu}\frac{Z_0{}^2}{k_0}\tau,$$

where Q is the amount of heat radiating from the surface of the earth. This is followed by a substitution $\theta=\theta_1+\theta_2$ (and a similar substitution for θ^*) which splits the problem into two parts, each of which is solved by classical methods. The result is an integral representation of the solution functions. A particular example is given in which $\Phi=\Phi^*=0$ and $F=0.25$.

C. G. Maple (Ames, Iowa).

Difference Equations, Special Functional Equations

Tanaka, Sen-ichiro. On asymptotic solutions of non-linear difference equations of the first order. I. Mem. Fac. Sci. Kyūsyū Univ. A. **7**, 107–127 (1953).

The author considers the solution of analytic difference equations of the form $y(x+1)=f(x,y(x))$ in the complex plane. To establish the existence of solutions he uses a fixed-point theorem, following a method used by Hukuhara in the theory of differential equations. By means of a unique-

ness theorem, he shows that certain formal solutions are actually asymptotic solutions. *R. Bellman.*

Fort, Tomlinson. The loaded vibrating net and resulting boundary-value problems for a partial difference equation of the second order. J. Math. Physics **33**, 94–104 (1954).

Let a rectangular net of weightless cords be loaded at each intersection point (x_i, y_j), $i=1, \cdots, m$; $j=1, \cdots, n$, with a particle of mass M. Let the tension in each string be assumed constant throughout its length. Let the displacements of the point masses be assumed small and perpendicular to the plane of the net at rest. These displacements u_{ij} satisfy the equations

$$(1)\qquad \frac{d^2u_{ij}}{dt^2}=\Delta_i\{b_{i-1,j}\Delta_i u_{i-1,j}\}+\Delta_j\{k_{i,j-1}\Delta_j u_{i,j-1}\},$$

where b_{ij}, k_{ij} depend on the mass M, the net mesh size, and the tensions in the strings at (x_i, y_j). A substitution $u_{ij}=e^{st}v_{ij}$, $\lambda=-s^2$ reduces the problem to that of solving mn linear algebraic equations

$$(2)\qquad \Delta_i\{b_{i-1,j}\Delta_i v_{i-1,j}\}+\Delta_j\{k_{i,j-1}\Delta_j v_{i,j-1}\}+\lambda v_{ij}=0$$

to which are added the boundary conditions

$$(3)\qquad v_{0j}=v_{i0}=v_{m+1,j}=v_{i,n+1}=0.$$

Combining (2) and (3) gives a system of mn linear homogeneous equations in $v_{11}, \cdots, v_{mn}$. The determinant of coefficients equated to zero is an equation in λ, the characteristic equation, with roots all real and positive. The problem is broadened by removing the requirement $v_{m,n+1}=0$ and solving an enlarged system of equations to obtain polynomials in λ:

$$v_{ij}=(-1)^{m+n-i-j+1}B\lambda^{mn-n-m+i+j-p-1}+A(\lambda),$$

where $B>0$, p is a non-negative integer not greater than $mn-m-n+1$ and independent of i and j, and $A(\lambda)$ represents terms of lesser degree. The polynomials satisfy the original boundary-value problem except possibly for the requirement $v_{m,n+1}=0$. The sign changes of the v_{ij} over the net, the behavior of the nodal lines, and the zeros of $v_{m,n+1}$ are studied as functions of λ. For the symmetrical case $m=n$, $b_{ij}=k_{ji}$ it is shown that $v_{ij}=v_{ji}$. For the factorable case $b_{ij}=B_iD_j$, $k_{ij}=M_iN_j$ a solution in the form $y_{ij}=s_ir_j$ is found by obtaining s_i and r_j as solutions of respective one-dimensional Sturm-Liouville systems. *P. E. Guenther.*

Wundt, Hermann. Über eine Funktionalgleichung aus der Wärmeleitung. Z. Angew. Math. Physik **5**, 172–175 (1954).

The construction of a parallel scale alignment chart for the determination of logarithmic mean temperatures from the equation $\Delta t_m=(\Delta t_1-\Delta t_2)/\ln(\Delta t_1/\Delta t_2)$ requires that the function $f(x)$ describing the ruling of the scales satisfy the functional equation

$$2f(x_m)=f(x_1)+f(x_2),\quad x_m=(x_1-x_2)/\ln(x_1/x_2).$$

It is shown that for every solution $f(x)$ the combination $Af(x)+B$ is a solution, that any continuous solution not identically constant is strictly monotone, and that for any differentiable solution not identically constant the derivative is strictly monotone. By straightforward manipulation of the original equation it is found that the most general differentiable solution is given by the formula $f(x)=A\int_1^x g(\xi)d\xi+B$, where

$$g(\xi)=(\ln\xi+(1/\xi)-1)/(\xi-\ln\xi-1).$$

P. E. Guenther (Cleveland, Ohio).

Bellman, Richard. Some problems in the theory of dynamic programming. Econometrica **22**, 37–48 (1954).

For any continuous functions g, h on $[0, x_0]$ with

$$g(0)=h(0)=0$$

and any constants a, b with $0\leq a$, $b<1$, the functional equation

$$(1)\qquad f(x)=\max_{0\leq y\leq x}[g(y)+h(x-y)+f(ay+b(x-y))]$$

has a unique bounded solution satisfying $f(0)=0$. If g, h are monotone, non-decreasing and convex, the maximizing y for a given x is either 0 or x. The problem of approximate solution of (1) is discussed. *D. Blackwell.*

Ghermănescu, M. Equations fonctionnelles linéaires. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 245–259 (1951). (Romanian. Russian and French summaries)

The author gives some general theorems on linear functional equations of a type which includes difference equations, and then specializes them to a class of difference equations which includes a number of special cases which have been considered previously. Let θ be a function with domain in p-space (e.g., the operator carrying x into $x+h$). Denote the kth iterate of θ by θ_k and write $f^{[n]}(x)=f\{\theta_n(x)\}$ [the author writes $f^n(x)$, which leads to ambiguity in some of his equations]. An "iterative constant" is a function invariant under θ (e.g., a function of period h if θ carries x into $x+h$). The most general equation considered is of the form

$$(1)\qquad f^{[n]}(x)+\phi_1(x)f^{[n-1]}(x)+\cdots+\phi_{n-1}(x)f^{[1]}(x)+\phi_n(x)f(x)=0.$$

A fundamental system of solutions of (1) is one consisting of functions connected by no linear relation with iterative constant coefficients, a condition expressible by the non-vanishing of a determinant analogous to the Wronskian. The principal theorem is then as follows. Equation (1), with the $\phi_i(x)$ given, has a solution if there are n functions $\lambda_i(x)$ each satisfying

$$(2)\qquad \lambda^{[n-1]}\lambda^{[n-2]}\cdots\lambda^{[1]}\lambda+\phi_1\lambda^{[n-2]}\lambda^{[n-3]}\cdots\lambda+\cdots+\phi_{n-1}\lambda+\phi_n=0$$

and (3) $\lambda_i^{[1]}-\lambda_j^{[1]}=\lambda_i-\lambda_j$ (so that the differences $\lambda_i-\lambda_j$ are iterative constants). The general solution is then a linear combination, with iterative constant coefficients, of the solutions of $f^{[1]}-\lambda_if=0$. If the $\phi_i(x)$ are not given, the solution has the same form, but with λ_i arbitrary functions subject to (3), and ϕ_i determined by

$$-\phi_1=\lambda_n^{[n-1]}+\lambda_{n-1}^{[n-2]}+\cdots+\lambda_1,$$

$$\phi_2=\sum_{i=0}^{n-2}\lambda_{n-i}^{[n-i-2]}(\lambda_{n-i-1}^{[n-i-2]}+\lambda_{n-i-2}^{[n-i-3]}+\cdots+\lambda_1),$$

etc. [It is not clear just what the term "general solution" means, but it is apparently relative to some restriction of the nature of continuity imposed on the solutions which are to be admitted. In the special cases treated later, the most general measurable solution is found.]

The author's principal aim is to study the "difference" case of (1) with iterative constant coefficients,

$$(4)\qquad f(x+ny, y)+\phi_1(x/y, y)f(x+(n-1)y, y)+\cdots+\phi_n(x/y, y)f(x, y)=0,$$

where $\phi(z, y)$ has period 1 in z; here y is a parameter, θ carries x into $x+y$, and the ϕ_i are usually not prescribed in advance. In this case each λ is to be an iterative constant and (2) reduces to

$$(5)\qquad \lambda^n+\phi_1\lambda^{n-1}+\cdots+\phi_n=0.$$

The general solution of (4) with given ϕ_i is then of the form $\sum P_i(x)(\lambda_i)^{x/y}$, where the λ_i are the solutions of (5) and $P_i(x)$ are polynomials in x with arbitrary iterative constant coefficients and degrees less than the order of multiplicity of λ_i as a solution of (5). If the ϕ_i are not given, the λ_i are arbitrary iterative constants and the ϕ_i are subject to

$$\frac{d^j}{d\lambda_i^j}[\lambda_i^n+\phi_1\lambda_i^{n-1}+\cdots+\phi_n]=0,\quad 0\leqq j\leqq n.$$

For example, there are two distinct types of solutions of

$$f(x+y, y)+\phi(x/y, y)f(x-y, y)=\psi(x/y, y)f(x, y),$$

where ϕ and ψ are initially subject only to the restriction that they have period 1 in the first variable.

Particular attention is paid to the special case where the ϕ_i depend only on y (and consequently the solutions f are independent of y). For example,

$$f(x+y)+\phi(y)f(x-y)=\psi(y)f(x)$$

has measurable (nontrivial) solutions f only if either $\phi(y)=e^{(a+b)y}$, $\psi(y)=e^{ay}+e^{by}$, when $f(x)=Ae^{ax}+Be^{bx}$, or $\phi(y)=e^{2ay}$, $\psi(y)=2e^{ay}$, when $f(x)=(Ax+B)e^{ax}$. The equations $f(x+y)+f(x-y)=f(x)\phi(x, y)$ and several similar ones are also discussed. *R. P. Boas, Jr.*

Integral Equations

Copping, J. Application of a theorem of Pólya to the solution of an infinite matrix equation. Pacific J. Math. **4**, 21–28 (1954).

This paper contains a novel application of a theorem of G. Pólya to establish the existence of solutions of the infinite matrix equation $AX-XB=C$. The theorem of Pólya is as follows [Comment. Math. Helv. 11, 234–252 (1939)]: In the infinite system of linear equations

$$(\mathrm{A})\qquad \sum_{j=1}^{\infty}a_{i,j}u_j=b_i\quad (i=1, 2, 3, \cdots),$$

where $\{b_i\}$ is an arbitrary sequence, let $(a_{i,j})$ satisfy the conditions: (i) the first row $a_{1,j}$ contains an infinity of nonzero elements; and

$$(\mathrm{ii})\qquad \liminf_{j\to\infty}[(|a_{1,j}|+|a_{2,j}|+\cdots+|a_{i-1,j}|)/|a_{i,j}|]=0$$

for every fixed $i\geqq 2$. Then there exists an infinite sequence $\{u_j\}$ satisfying (A), such that all the left sides are absolutely convergent.

The author assumes that in the above equation

$$AX-XB=C,$$

B and C are arbitrary given matrices, and A is a given matrix satisfying (i) and (ii) of Pólya's theorem. He then shows that the equation has an infinity of solutions; in particular, it has an infinity of solutions each of which is a lower semi (triangular) matrix whose principal diagonal elements are zero. To prove this, the author first gives (as his principal tool) the following result: Suppose A satisfies (i) and (ii) of Pólya's theorem, and B and C are arbitrary given infinite matrices. Let the set of linear equations

$$(\mathrm{B})\qquad \sum_{r,s}(\delta_{k,s}a_{n,r}-\delta_{n,r}b_{s,k})x_{r,s}=c_{n,k}$$

be written in the order

$$(n=1, k=1),\ (n=2, k=1),\ (n=1, k=2),\ \cdots,$$

where $\sum_{r,s}$ denotes the "Cauchy sum" $(r=1, s=1)$, $(r=2, s=1)$, $(r=1, s=2)$, $\cdots$. Then the matrix of the system of equations (B) satisfies the conditions (i) and (ii) of Pólya's theorem.

The following result is next given. Suppose A satisfies (i) and (ii) of Pólya's theorem, and let $P=(p_{n,k})$ be any given matrix such that AP and PB exist. Then there is an infinity of solutions of the equation $AX-XB=C$, for each of which $x_{n,k}=p_{n,k}$ for all $k\geqq n$. Finally, two negative results are proved concerning the particular cases of $AX-XB=C$ which occur in quantum mechanics, viz., $AX-XD=0$ and $AX-XA=I$, where D is a diagonal matrix, and I is the unit matrix. It is shown that the methods employed in this paper fail to give solutions Y of $AX-XD=0$ such that $YDY^{-1}=A$; they do, however, give solutions Y such that $Y^{-1}AY=D$, but these solutions clearly cannot be such that Y, Y^{-1}, and A should belong to the same "associative field" [see the reviewer's Infinite matrices and sequence spaces, Macmillan, London, 1950, pp. 9, 26; these Rev. 12, 694]. It is finally shown that if A belongs to a "field with an associative bound" [loc. cit., p. 27], then no solution of $AX-XA=I$ belongs to the same field. *R. G. Cooke.*

Satō, Tokui. Sur l'équation intégrale non linéaire de Volterra. Compositio Math. 11, 271–290 (1953).

The integral equation under consideration is

$$(1)\qquad u(x)=f(x)+\int_a^x K(x, t, u(t))dt,$$

where $f(x)$ is continuous on $a\leqq x\leqq a+r$, $K(x, t, u)$ continuous on $a\leqq t\leqq x\leqq a+r$, $|u(x)-f(x)|\leqq\rho$. The special case where K is independent of x is essentially a differential equation and the results obtained are extensions of known results for differential equations [see Kamke, Differentialgleichungen reeller Funktionen, Akademische Verlagsgesellschaft, Leipzig, 1930, pp. 59–99]. Thus we have the usual existence theorem and a uniqueness theorem based on a Lipschitz condition for K in u. Basic for later considerations is a comparison theorem which asserts that if $\bar{u}$ satisfies an integral equation like (1) for $\bar{f}$ and $\bar{K}$ with $f(a)\leqq\bar{f}(a)$, $f(\bar{x})-f(x)\leqq\bar{f}(\bar{x})-f(x)$ for $\bar{x}>x$, $K(x, t, u)<\bar{K}(x, t, u)$ in x, t, u and

$$\lambda K(\bar{x}, t, u)+\mu(K(\bar{x}, t, u)-K(x, t, u))\leqq\lambda\bar{K}(\bar{x}, t, \bar{u})+\mu(\bar{K}(\bar{x}, t, \bar{u})-\bar{K}(x, t, \bar{u})),$$

for λ, μ fixed with $0\leqq\lambda$, $\mu\leqq 1$, $\lambda+\mu=1$ and $x<\bar{x}$, $u<\bar{u}$, then for any solution $u(x)$ of (1) we have $u(x)<\bar{u}(x)$ on $a<x\leqq a+r$. This leads to the existence of maximal and minimal solutions of (1) and solutions of corresponding inequalities, as well as a uniqueness theorem based on the existence for K of a function $\bar{K}$ which in addition to the monotone conditions above for $\lambda=1$, $\mu=0$ satisfies an inequality of the form $|K(x,t,u)-K(x,t,\bar{u})|\leqq\bar{K}(x,t,|u-\bar{u}|)$ with $\bar{u}(x)=\int_a^x\bar{K}(x, t, \bar{u}(t))dt$ having as unique solution $\bar{u}(x)=0$. The results are extended to systems of differential equations, and an extension of a theorem of Kneser [S.-B. Preuss. Akad. Wiss. 1923, 171–176] that the intersection by a plane $x=c$ with $a<c\leqq a+r$ of the solutions of (1) form a continuum in u-space is proved. *T. H. Hildebrandt.*

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Pirl, Udo. Positive Lösungen einer nichtlinearen Integralgleichung. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. ~~100, no. 8~~, 44 pp. (1953).

Consider the non-linear integral equation for the unknown function $y=y(s)$

$$(1)\qquad P[y]+K[y]=f(s),$$

where

$$(2)\qquad P[y]=\sum_{n=1}^{\infty}a_n(s)y^n(s)$$

$$(3)\quad K[y]=\sum_{n=1}^{\infty}\sum_{\alpha+\alpha_1+\cdots+\alpha_\nu\leq n}\int_a^b\cdots\int_a^b K_{\alpha\alpha_1\cdots\alpha_\nu}(s,t_1,\cdots,t_\nu)\times y^{\alpha}(s)y^{\alpha_1}(t_1)\cdots y^{\alpha_\nu}(t_\nu)dt_1\cdots dt_\nu.$$

Such an equation had been treated by W. Schmeidler [Math. Nachr. **8**, 31–40 (1952); these Rev. **14**, 382] in the case that the sums in (2) and (3) contain only a finite number of terms. The present paper treats the general case of infinite series. It is, however, supposed that the index α in (3) is zero throughout. In addition to continuity and convergence hypotheses concerning (2) and (3), the assumptions

$$(4)\qquad \partial P/\partial y>0 \text{ for } y>0 \quad\text{and}\quad f(s)>0 \text{ in } a\leq s\leq b$$

are made. First the case (5) $K_{\alpha\alpha_1\cdots\alpha_n}\leq 0$ is treated and a positive solution $y(s)$ is sought under the additional assumption that $K[y]\not\equiv 0$ and that there exists a continuous $y_0(s)$ and $d>0$ for which

$$(6)\quad P(y_0)>f(s)-K(y_0),\quad 0<y_0(s)\leq d \quad\text{for}\quad a\leq s\leq b.$$

It is shown that in this case the approximation procedure used by Schmeidler (in the paper cited above) can still be applied and yields a monotone decreasing sequence of bounded continuous functions whose limit $y(s)$ is a positive solution of (1).

In the general case the decomposition

$$(7)\qquad M_{\alpha\alpha_1\cdots\alpha_n}=\max_{a\leq s\leq b}[K_{\alpha\alpha_1,\cdots,\alpha_n},0]$$

$$(8)\quad L_{\alpha\alpha_1\cdots\alpha_n}=M_{\alpha_1\cdots\alpha_n}-K_{\alpha_1\cdots\alpha_n},\quad K[y]=M[y]-L[y]$$

is used where $M[y]$ and $L[y]$ are the series obtained if the coefficients $K_{\alpha\alpha_1\cdots\alpha_n}$ in (3) are replaced by (7) and (8) respectively. Then (again according to Schmeidler) instead of (1) the following auxiliary equation is treated first:

$$(9)\qquad P[y]=f-\lambda+L[y],\quad 0\leq\lambda<f(s).$$

(9) can be treated by the procedure described above which yields for each λ a positive solution $y(s,\lambda)$ of (9) which is continuous in s but not necessarily in λ. The following condition is shown to be sufficient for the continuity of $y(s,\lambda)$ in λ. Let $y^*(s)$ be the solution of $P[y]=f(s)-\lambda$ and let $y_k(s,\lambda)$ $(k=1,\cdots)$ be the approximating sequence (referred to above) of (9); let $\omega^*=\min y^*(s,\lambda)$, $\Omega_k=\max y_k(s,\lambda)$ and $\rho_k>0$ the inf of $\partial P/\partial y$ in $\omega^*\leq y\leq\Omega_k$. Finally, let

$$A_n=\max\sum_{\alpha_1+\alpha_n=n}\int_a^b\cdots\int_a^b L_{\alpha_1\alpha_n}(s,t_1,\cdots,t_n)dt_1\cdots dt_n$$

and $F(\Omega)=\sum A_n\Omega^n$. Then the sufficiency condition in question is that $F'(\Omega_k)/\rho_k<1$ for some k. In case $y(s,\lambda)$ is continuous in λ, $y(s,\lambda_0)$ will be a solution of (1) if λ_0 is a solution of $M[y(s,\lambda_0)]=\lambda_0$.

In addition to the described procedure which approximates from above, the author devises another procedure which approximates from below. Finally, the case is investigated where in (3) the index α is not necessarily zero but all kernels are non-negative. *E. H. Rothe.*

Rutic'kiĭ, Ya. B. On a theorem of M. M. Nazarov. Dopovidi Akad. Nauk Ukrain. RSR **1952**, 91–95 (1952). (Ukrainian. Russian summary)

The author notices that equation

$$(1)\qquad \varphi(x)+\int_0^1\int_0^1 K(x,s)h(s,t;\varphi(t))dsdt=0$$

is a special case of equation

$$(2)\quad \varphi(y,x)+\int_0^1\int_0^1 K(y,x,s,t)h(s,t;\varphi(s,t))dsdt=0$$

and using results known for equation (2) formulates an existence theorem for equation (1). *M. Golomb.*

Elliott, Joanne. The boundary value problems and semi-groups associated with certain integro-differential operators. Trans. Amer. Math. Soc. **76**, 300–331 (1954).

This is a detailed study of the initial and boundary-value problems for the functional equations

$$u_t(x,t)=\pi^{-1}P\int_{-1}^{1}\frac{u_\xi(\xi,t)}{\xi-x}d\xi\equiv\Omega u,$$

$$v_t(x,t)=\pi^{-1}\frac{d}{dx}P\int_{-1}^{1}\frac{v(\xi,t)}{\xi-x}d\xi\equiv\Omega^*v,$$

where the integrals are to be taken in the sense of the Cauchy principal value for $-1<x<+1$ and as essential limits of such integrals for $x\to\pm1$ at the endpoints. Initial values and solutions are taken in $C[-1,1]$ for the first and in $L(-1,1)$ for the second equation, though for the more refined parts of the theory it is necessary to consider a product space of $L(-1,1)$ with the real line or with the real plane. Since each of the homogeneous equations

$$\Omega y-\lambda y=0,\quad \Omega^*s-\lambda s=0$$

has two linearly independent solutions it is necessary to introduce suitable boundary conditions. These are chosen so that Ω and Ω^* become the infinitesimal generators of semi-groups of positive contraction operators which, when acting on the initial values, give the desired solutions. The resulting conditions are closely analogous to W. Feller's conditions for the diffusion equations

$$u_t(x,t)=u_{xx}(x,t)-\frac{x}{1-x^2}u(x,t),$$

$$v_t(x,t)=\frac{\partial}{\partial x}\left\{v_x(x,t)-\frac{x}{1-x^2}v(x,t)\right\}$$

with $x=\pm1$ as "regular boundaries" [Ann. of Math. (2) **55**, 468–519 (1952); these Rev. **13**, 948]. The resolvent equation $\lambda f-\Omega f=h$ is equivalent to the integral equation.

$$f(x)+\lambda\int_{-1}^{1}K(x,y)f(y)dy=\int_{-1}^{1}K(x,y)h(y)dy+A\arcsin x+B$$

where

$$K(x,y)=\frac{1}{2\pi}\log\left\{\frac{1-xy+[(1-x^2)(1-y^2)]^{1/2}}{1-xy-[(1-x^2)(1-y^2)]^{1/2}}\right\},$$

the solution of which is

$$f(x)=\int_{-1}^{1}\Gamma(x,y;\lambda)h(y)dy+A\phi_1(x)+B\phi_2(x).$$

Here $\Gamma(x,y;\lambda)$ is the resolvent kernel of $K(x,y)$, it is symmetric, continuous except for $x=y$, and non-negative.

Further,

$$\phi_\nu(x)=\psi_\nu(x)-\lambda\int_{-1}^{1}\Gamma(x,y;\lambda)\psi_\nu(y)dy,$$

$$\psi_1(x)=\arcsin x,\quad \psi_2(x)=\tfrac{1}{2}$$

are the solutions of the homogeneous equation.

In the simplest case (absorbing barrier in Feller's terminology) $A=B=0$ and $\lambda R(\lambda;\Omega)[h]=f$ is a positive contraction operator on the space $C_{00}[-1,1]$ of continuous functions vanishing at both ends of the interval. By the Hille-Yosida theorem Ω then generates a semi-group of positive contraction operators in this space. Explicit representations of these operators are not given, but are easily obtained from the author's expression for the resolvent. The author also gives a detailed analysis of the case in which there is a single absorbing barrier at $x=-1$ while $x=1$ is free. Here the resolvent is given by

$$f(x)=\int_{-1}^{1}\Gamma(x,y;\lambda)h(y)dy+Q^*(h)\lambda^{-1}[\phi_2(x)+\pi^{-1}\phi_1(x)]$$

where Q^* is a positive functional with $\|Q^*\|\leq 1$. The general form of such a functional is found as well as the subspace spanned by the resolvent. The formulation of the resulting boundary condition at the free boundary leads to similar complications as in the differential equation case. The adjoint equation, its resolvent and the various possible lateral conditions are also examined in detail. The extensions to two free boundaries are briefly indicated.

E. Hille (New Haven, Conn.).

Vasilache, S. Sur un système d'équations intégro-différentielles rencontrées dans différents problèmes de sciences techniques. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 311–317 (1951). (Romanian. Russian and French summaries)

The author solves the system

$$aT_\theta(x,\theta)+bT_x(x,\theta)-t(\theta)+T=0,$$

$$ct'(\theta)-lt(\theta)=-\int_0^l T(\xi,\theta)d\xi,\quad T(0,\theta)=T_0,\quad t(0)=t_0$$

by constructing the integral equation satisfied by T, and solving it by successive approximations. *A. Erdélyi.*

Bouvier, Pierre B. Les équations intégrales de Milne pour une atmosphère parfaitement diffusante. Arch. Sci. Soc. Phys. Hist. Nat. Genève **6**, 262–268 (1953).

Some formal relations analogous to those known in the case of conservative isotropic scattering in the theory of radiative transfer are derived for the case of Rayleigh scattering. *S. Chandrasekhar* (Williams Bay, Wis.).

Ramakrishnan, Alladi, and Mathews, P. M. On the solution of an integral equation of Chandrasekhar and Münch. Astrophys. J. **119**, 81–90 (1954).

Chandrasekhar and Münch [same J. **112**, 380–392 (1950); these Rev. **12**, 644] in their theory of the fluctuations in brightness of the Milky Way have derived the following integral equation

$$(1)\qquad \frac{\partial g(u,\xi)}{\partial\xi}=-g(u,\xi)-\frac{\partial g(u,\xi)}{\partial u}+\int_{1/q}^{1}g(u/q,\xi)\psi(q)\frac{dq}{q};$$

and in the case $\psi(q')=\delta(q-q')$ where q is an assigned constant less than one and δ denotes Dirac's delta function, they obtained the solution in the form [ibid. **114**, 110–122 (1951); these Rev. **13**, 249]

$$(2)\qquad g(u,\xi)=Ke^{-\xi}\sum_{k=0}^{l-1}\frac{Q_k}{q^k}\sum_{r=0}^{\infty}\frac{Q_r}{q^{(k-1)r}}\exp\left[q^r\left(\xi-\frac{u}{q^k}\right)\right]+e^{-\xi}\delta(u-\xi)\quad (\xi q^l<u<\xi q^{l-1})$$

and

$$(2')\qquad \frac{1}{K}=\prod_{j=1}^{\infty}(1-q^j)\quad\text{and}\quad Q_k=(-1)^k\prod_{j=1}^{k}\frac{q_j}{1-q^j}.$$

In this paper the authors obtain the same solution by a different method. By applying a Mellin transformation

$$(3)\qquad p(s,\xi)=\int_0^\infty g(u,\xi)u^s du,$$

where s is a complex variable, the authors first transform (1) into

$$(4)\qquad \frac{\partial p(s,\xi)}{\partial\xi}=-p(s,\xi)+q_s p(s,\xi)+sp(s-1,\xi),$$

where q_s is the sth moment of q $(=\int_0^\infty q^s\psi(q)dq)$. If $s=n$ is an integer, the solution of (4) is

$$(5)\qquad p(n,\xi)=n!\sum_{k=0}^{n}\frac{e^{-(1-q_k)\xi}}{\prod_{j=0,\,j\neq k}^{j=n}(q_k-q_j)}.$$

The authors show how this expression can be given a meaning quite generally even when n is not an integer (this is the main point of the paper); and once this has been done, the required solution for g can be obtained by taking the inverse Mellin transformation. In this way the authors obtain the solutions for all the known cases.

S. Chandrasekhar (Williams Bay, Wis.).

Bahrah, L. D. On the solution of the integral equation of a linear antenna. Doklady Akad. Nauk SSSR (N.S.) **92**, 755–758 (1953). (Russian)

The author solves the integral equation

$$F(\phi)=\sin\phi\int_{-a}^{a}I(z)e^{ikz\cos\phi}dz$$

by putting $z=a\cos t$ whereupon a symmetric kernel appears whose characteristic functions are Mathieu functions.

A. Erdélyi (Pasadena, Calif.).

Byrd, Paul F. An integral equation occurring in the theory of a slender quasi-axisymmetrical body. J. Aeronaut. Sci. **21**, 351 (1954).

The author proves that

$$g(x,\psi)=\int_{-1}^{1}\frac{H(x,\zeta)d\zeta}{\psi-\zeta},\quad |\psi|>1,$$

has the solution

$$H(x,\zeta)=\frac{1}{\pi a}\operatorname{Im}\{[\zeta+i(1-\zeta^2)^{1/2}]g(x,\zeta)\}.$$

J. W. Miles (Los Angeles, Calif.).

Functional Analysis, Ergodic Theory

Monna, A. F. Sur le théorème de Hahn-Banach. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 9–16 (1954).

The author shows that Banach's "algebraic" version of the Hahn-Banach theorem implies Dieudonné's "geometric"

version, which deals with open convex sets and linear varieties in a topological linear space. [Reviewer's note: This was already shown by S. Mazur [Studia Math. 4, 70–84 (1933)], although Mazur's result is stated only for normed spaces.] *E. Michael* (Seattle, Wash.).

Ono, Takashi. On the extension property of normed spaces over fields with non-archimedean valuations. J. Math. Soc. Japan 5, 1–5 (1953).

The author proves that a non-Archimedean normed linear space S over a complete discrete valued field k possesses the Hahn-Banach extension property. The proof utilizes an idea of L. Nachbin [Trans. Amer. Math. Soc. 68, 28–46 (1950); these Rev. 11, 369]. The author states that the papers by Ingleton [Proc. Cambridge Philos. Soc. 48, 41–45 (1952); these Rev. 13, 659], Monna [Indagationes Math. 8, 682–689 (1946); these Rev. 9, 43], and Cohen [ibid. 10, 244–249 (1948); these Rev. 10, 48] were not available to him when he prepared the paper under review.
G. K. Kalisch (Minneapolis, Minn.).

Nevanlinna, Rolf. Über metrische lineare Räume. IV. Zur Theorie der Unterräume. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 163, 16 pp. (1954).

Continuing his earlier work [same Ann. nos. 108, 113, 115 (1952); Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 160–168 (1952); these Rev. 14, 287, 658], the author discusses the classical problem of projections in his generalized Hilbert spaces in which the inner product is not necessarily positive definite. Let $Q(x, y)$ be the generalized inner product $(Q(x, x)=Q(x))$ and $H(x, y)$ a complete majorant positive-definite inner product $(H(x, x)=H(x))$ for which $|Q(x, y)|^2 \leqq H(x)H(y)$. Write

$$Q(x, y)=\int_{-1}^{1} \lambda dH(E_\lambda(x), y)$$

and let $E_1=\int_{0+}^{1} dE_\lambda$, $E_2=\int_{-1}^{0-} dE_\lambda$, $Q_i(x, y)=(-1)^{i+1}Q(E_i x, y)$. Then $Q=Q_1-Q_2$, and, among all positive definite $K(x, y)$ which majorize $Q(x, y)$, the form $P(x, y)=Q_1(x, y)+Q_2(x, y)$ is minimal, i.e., $(P(x, y))^2 \leqq K(x)K(y)$ for all majorizing $K(x, y)$. Then P defines a complete Hilbert space if and only if there is an $\epsilon>0$, for which $dE_\lambda \equiv 0$ for all $|\lambda|<\epsilon$.

Let U be a linear manifold closed with respect to H. For a given x, the author finds that a normal from x to U relative to Q can exist if and only if $Q(x, u_0)\equiv 0$ for all $u_0 \in U_0=\{y \mid y \in U, Q(y, v)=0, \text{ all } v \in u\}$ and

$$-\infty<\inf_{U_1} Q(x-u_1), \quad \sup_{U_2} Q(a-u_2)<\infty,$$

where U_i are the ranges of the projections E_i in U. The technique involves the Zaremba-Riesz technique of proving that a minimizing sequence is convergent. *B. Gelbaum.*

Koshi, Shozo. On Weierstrass-Stone's theorem. J. Math. Soc. Japan 5, 351–352 (1953).

The author obtains an elementary consequence of the Stone-Weierstrass theorem. *E. Michael.*

***Pellegrino, F., e Tomacelli, L. Determinazione degli zeri di una classe di funzionali analitici lineari.** Società Italiana per il Progresso delle Scienze, XLII riunione, Roma, 1949, Relazioni, Vol. primo, pp. 193–202. Società Italiana per il Progresso delle Scienze, Roma, 1951.

Les auteurs commencent par résoudre l'équation fonctionnelle en $y(t)$, $f(t)=y(t)+v(t)y(t^{-1})$, et ils appliquent ensuite ce résultat à l'étude de l'équation $F_t[y(t)]=0$, où F est une fonctionnelle linéaire continue, dont l'indicatrice $u(\alpha)=F_t[(\alpha-t)^{-1}]$ vérifie la condition $u(\alpha)=\alpha^{-1}u(\alpha^{-1})$. Soit $\mathfrak{H}$ l'ensemble des zéros de F; si l'on pose

$$\phi_t[y(t)]=y(t)-t^{-1}y(t^{-1}),$$

on a $y \in \mathfrak{H}$, si et seulement si $\phi(y) \in \mathfrak{H}$, ce qui ramène la recherche des zéros de F à ceux de la forme $\phi[y]$.
J. Sebastião e Silva (Lisbonne).

***Fantappiè, Luigi. Gli operatori lineari permutabili con un gruppo continuo.** Società Italiana per il Progresso delle Scienze, XLII riunione, Roma, 1949, Relazioni, Vol. primo, pp. 163–165. Società Italiana per il Progresso delle Scienze, Roma, 1951.

Soit $\bar{\mathbf{C}}^n$ l'espace projectif à n dimensions complexes et soit G un groupe continu de transformations à r paramètres, d'équations $\bar{\mathbf{x}}=g(\mathbf{x}, \mathbf{h})$, avec $\mathbf{x}, \bar{\mathbf{x}} \in \bar{\mathbf{C}}^n$, $\mathbf{h} \in \bar{\mathbf{C}}^r$. Étant donnée une application analytique, $K(\varphi)=\psi$, d'une région $\mathfrak{R}$ de l'espace fonctionnel $\mathfrak{S}^{(n)}$ dans $\mathfrak{S}^{(n)}$, on dit que K est permutable avec G, si l'on a $K_{\mathbf{x}}\varphi[g(\mathbf{x}, \mathbf{h})]=\psi[g(\mathbf{x}, \mathbf{h})]$, pour chaque $\mathbf{h} \in \bar{\mathbf{C}}^r$ et chaque $\varphi \in \mathfrak{R}$. En se bornant au cas où G est simplement transitif et $n=r$, l'auteur énonce les résultats suivants: (a) les opérateurs linéaires K permutables avec G peuvent toujours se mettre sous la forme $K\varphi=F_{\mathbf{h}}\varphi[g(\mathbf{x}, \mathbf{h})]$, où F est une fonctionnelle à valeurs numériques, agissant sur des fonctions de $\mathbf{h}$; (b) pour qu'un opérateur de cette forme soit permutable avec G, il suffit que l'indicatrice projective $p(\mathbf{h})=p(h_1, \cdots, h_n)$ de la fonctionnelle F soit invariante pour toutes les transformations du groupe adjoint.
J. Sebastião e Silva (Lisbonne).

***Varsano, S. Le indicatrici dei funzionali lineari del ciclo chiuso delle funzioni di più variabili.** Società Italiana per il Progresso delle Scienze, XLII riunione, Roma, 1949, Relazioni, Vol. primo, pp. 181–183. Società Italiana per il Progresso delle Scienze, Roma, 1951.

Il s'agit d'un exposé de résultats qui ont déjà été développés dans un autre travail [Univ. Roma 1st. Naz. Alta Mat. Rend. Mat. e Appl. (5) 9, 123–135 (1950); ces Rev. 15, 235]. *J. Sebastião e Silva* (Lisbonne).

Ważewski, T. Une généralisation des théorèmes sur les accroissements finis au cas des espaces de Banach et application à la généralisation du théorème de l'Hôpital. Ann. Soc. Polon. Math. 24 (1951), no. 2, 132–147 (1954).

This paper is concerned with reformulations and generalizations of the law of the mean, of a type indicated by the following theorem. Suppose f is real-valued, continuous and strictly monotonic on an open interval of the real axis. Let Φ be continuous on Δ, with values in a Banach space D. Let V be a closed convex set in D, and A an at most countably infinite subset of Δ. Let

$$I(p, q)=\frac{\Phi(q)-\Phi(p)}{f(q)-f(p)},$$

and let $\rho(u, V)$ be the distance from u to V (where $u \in D$). Then

$$\liminf_{h\to 0^+} \rho(I(t, t+h), V)=0$$

for $t \in \Delta-A$ implies $I(p, q) \in V$ when $p, q \in \Delta$, $p\neq q$. This theorem contains as a special case a generalization of a fundamental theorem in the work on contingents and paratingents by Marchaud [Compositio Math. 3, 89–127 (1936)] and Zaremba [Bull. Sci. Math. (2) 60, 139–160 (1936)]. The theorem can be expressed in terms of the concept of a

"bout différential," which is a generalization of the concept of a "dérivée contingentielle." For a fixed t the "bout différential" is a point set in the space D, defined in terms of the values of the quotient $I(s, t)$ for values of s near t.

The paper also contains generalizations of l'Hospital's rule for $I(p, q)$ based on a previous paper by the same author [Prace Mat.-Fiz. **47**, 117–128 (1949); these Rev. **11**, 585].

A. E. Taylor (Nancy).

Alexiewicz, A. On a theorem of Ważewski. Ann. Soc. Polon. Math. **24** (1951), no. 2, 129–131 (1954).

For convenience we adopt the notation of the preceding review, except that Φ is now assumed to be weakly continuous. Alexiewicz draws the same conclusion as Ważewski, namely that $I(p, q) \in V$ when $p, q \in \Delta$, $p \neq q$, from the hypothesis that for each $t \in \Delta - A$ there is a sequence of elements $y_n \in V$ and real positive h_n, $h_n \to 0$, such that $I(t, t+h_n) - y_n$ converges weakly to 0 as $n \to \infty$. The proof depends on Mazur's theorem about the existence of a hyperplane separating V and a point not in V. The author also exhibits a function $x(t)$ from $[0, 1]$ to a separable Banach space X, satisfying a Lipschitz condition, and yet such that, for each t, each $y \in X$, and each null sequence $\{h_n\}$, $[x(t+h_n) - x(t)]/h_n - y$ fails to converge weakly to zero. Thus the weak contingential derivative of $x(t)$ is an empty set for each t. This example answers a question raised earlier by Ważewski. *A. E. Taylor.*

Phillips, R. S. An inversion formula for Laplace transforms and semi-groups of linear operators. Ann. of Math. (2) **59**, 325–356 (1954).

The author discusses the conditions that a closed linear operator be the infinitesimal generator (i.g.) of a semi-group of bounded linear operators (b.l.o.) $T(s)$, $s>0$, Abel summable to the identity at $s=0$. It is proved, in section 2, that $F(\lambda)$ is the Laplace transform of a measurable submultiplicative function $(0 \leqq \varphi(s_1+s_2) \leqq \varphi(s_1)\varphi(s_2))$ integrable near $s=0$, if the following conditions are satisfied: (i) there exists a real ω such that $F(\lambda)$ is completely monotonic in $[\omega, \infty)$, (ii) $\lim_{\lambda\to\infty} F(\lambda)=0$ and (iii) $G(k+m+1, \lambda) \leqq G(k, \lambda)G(m, \lambda)$ for $G(n, \lambda) = (-1)^n (n!)^{-1} F^{(n)}(\lambda)$, and for integers k, $m \geqq 0$. This result is applied, in section 3, to the class of semi-groups $T(s)$ satisfying $\int_0^1 \|T(s)x\| ds < \infty$ for each x. For those $T(s)$ strongly Abel summable to the identity at $s=0$, the i.g. A defined by strong $\lim_{h\downarrow 0} h^{-1}(T(h)-I)x$ is not in general closed. The Laplace transform of such $T(s)$ is shown to be equal to the resolvent of $\bar{A}$, the smallest closed extension of A. A necessary and sufficient condition that $A=\bar{A}$ is that $T(s)$ be strongly Cesàro summable to the identity at $s=0$. In section 4, similar results are obtained under the further hypothesis $\int_0^1 \|T(s)\| ds$, specializing to give results concerning semi-group theory due to Hille, the reviewer and the author. In the final section is discussed the perturbation, by a b.l.o., of the i.g. of semi-groups of the above type. The method and the results are extensions of the author's previous paper [Trans. Amer. Math. Soc. **74**, 199–221 (1953); these Rev. **14**, 882]. *K. Yosida.*

Hille, Einar. Une généralisation du problème de Cauchy. Ann. Inst. Fourier Grenoble **4** (1952), 31–48 (1954).

Let U be a linear operator defined on a dense domain $D(U)$ of a complex Banach space X to X. The author discusses the "abstract Cauchy's problem in X":

$$(1)\quad y^{(n)}(t) = U^n y(t),\ t>0,\ \text{strong} \lim_{t\downarrow 0} y^{(k)}(t) = y \quad (k=0, 1, \cdots, n-1),$$

the derivatives being taken in the strong sense. Theorem 3 gives a uniqueness criterion: Let U^n be closed and such that the eigenvalues of U^n are not dense in the half plane $R(\lambda) > \lambda_0$. Then, for each system $(y_0, \cdots, y_{n-1})$, there exists at most one solution of (1) satisfying

$$(2)\qquad \limsup_{t\to\infty} t^{-1} \log \|y(t)\| < \infty.$$

Besides the existence theorem (Theorem 2) directly relying upon semi-group theory (due to the author and the reviewer), several existence theorems are obtained. For example, Theorem 5 gives an "abstract Stokes' formula": Let $n=2$ and let U be the infinitesimal generator of a group $T(t; U)$, $-\infty < t < \infty$, of bounded linear operator strongly continuous at $t=0$. Then, for $y_0 \in D(U^2)$ and $y_1 \in D(U) \cap \text{range}(U)$, the solution of (1) satisfying (2) is unique and is given by

$$y(t) = \tfrac{1}{2}\{T(t; U)(y_0+z_1) + T(-t; U)(y_0 - z_1)\}$$

with $Uz_1 = y_1$. These results are illustrated by typical equations: Cauchy-Riemann equation, wave equation, Laplace equation, $\partial^2 y/\partial t^2 = \partial^2 y/\partial x^2$ and $\partial^4 y/\partial t^4 = \partial^4 y/\partial x^4$.

K. Yosida (Princeton, N. J.).

Hille, Einar. Le problème abstrait de Cauchy. Univ. e Politecnico Torino. Rend. Sem. Mat. **12**, 95–103 (1953).

The author presents a detailed discussion of the abstract Cauchy problem, including for completeness a few results of two earlier papers [E. Hille, Ann. Soc. Polon. Math. **25**, 56–68 (1953); these Rev. **15**, 39; and the paper reviewed above]. Let X be a Banach space and U a closed linear operator with domain and range in X. Then given $y_0 \in X$, the Cauchy problem (PAC) consists of finding a function $y(t)$ on $(0, \infty)$ to X such that (i) $y(t)$ and $y'(t)$ belong to X for $t>0$; (ii) $y'(t) = U[y(t)]$ for $t>0$; and (iii) $\lim_{t\downarrow 0} \|y(t) - y_0\| = 0$. A solution $y(t) \not\equiv \theta$ is said to be a null solution if $y_0 = \theta$. A function $y(t)$ is said to be of type ω if $\limsup_{t\to\infty} t^{-1} \log \|y(t)\| = \omega$. The basic uniqueness theorem reads: If the proper values of U are not dense in any right half-plane, then the PAC has at most one solution of finite type for each $y_0 \in X$. In the opposite direction, the author shows that a necessary and sufficient condition that the PAC admit a null solution of type ω is that the equation $U[x(\lambda)] = \lambda x(\lambda)$, $R(\lambda) > \omega$, admit a solution $x(\lambda)$, $\not\equiv \theta$, holomorphic in $R(\lambda) > \omega$, and bounded in each half-plane $R(\lambda) \geqq \omega + \epsilon$, $\epsilon > 0$. Various connections of the PAC with the theory of semi-groups are considered. It is shown that a closed linear operator U with dense domain is the infinitesimal generator of a semi-group strongly continuous to the identity at $t=0$ if and only if there exists a family of linear bounded operators $[T(t); t \geqq 0]$ of finite type and strongly continuous for $t \geqq 0$ with $T(0) = I$ such that

$$R(\lambda; U)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt, \quad x \in X.$$

The paper concludes with a discussion of explosive solutions of the PAC which occur in an abstract L-space when the equation $U[x(\lambda)] = \lambda x(\lambda)$ has a solution $x(\lambda)$ which (i) is an entire function of order α, $0 \leqq \alpha \leqq 1$, and of minimal type if $\alpha = 1$; (ii) $x(\lambda)$ in the positive cone of X for $\lambda \geqq 0$; and (iii) $\|x(\lambda)\|$ non-$\in L(0, \infty)$. On p. 101, lines -3 and -5, and on p. 102, lines -7 and -9, the symbol ϵ with a stroke through it should be replaced by ϵ. *R. S. Phillips.*

Robison, Gerson B. Invariant integrals over a class of Banach spaces. Pacific J. Math. 4, 123–150 (1954).

Soient S un espace topologique, B l'espace de Banach des fonctions réelles bornées sur S, X un sous-espace fermé de B contenant 1, G un semi-groupe d'opérateurs linéaires continus dans X. Il s'agit de trouver une forme linéaire continue x^* sur X telle que $x^*(1)=1$, $x^*(x)\geqq 0$ pour $x\geqq 0$, $x^*(x)>0$ pour $x\geqq 0$, $x\neq 0$, et invariante par G. L'auteur trouve des conditions variées pour l'existence ou l'unicité d'une telle forme, en donnant le maximum de généralité aux raisonnements de von Neumann pour la mesure de Haar des groupes compacts. *J. Dixmier* (Paris).

Sikorski, R. On multiplication of determinants in Banach spaces. Bull. Acad. Polon. Sci. Cl. III. 1, 219–221 (1953).

The note announces a theorem for products of Fredholm determinants of linear functional transformations on linear normed complete spaces. The basis is essentially that of Lezanski [Studia Math. 13, 244–276 (1953); these Rev. 15, 535] except that the linear transformation T is replaced by a transformation T_F derived from the linear functional F on $\mathfrak{R}$; the Fredholm determinant $D(F)$, depending only on F, is similar to that used by A. F. Ruston [Proc. London Math. Soc. (3) 1, 327–384 (1951), p. 362; these Rev. 13, 468] and the product $F\cdot G$ is subject to the condition $F\cdot G(K)=G(KT_F)$ for every K of $\mathfrak{R}$. *T. H. Hildebrandt.*

Nemyckiĭ, V. V. The structure of the spectrum of nonlinear completely continuous operators. Mat. Sbornik N.S. 33(75), 545–558 (1953). (Russian)

This is a more detailed presentation of results published earlier [Doklady Akad. Nauk SSSR (N.S.) 80, 161–163 (1951); these Rev. 13, 356]. There are a few additions centering about the following theorem: Suppose $A(\varphi)$ is completely continuous and has the Fréchet differential $L(\varphi, h)$; suppose λ_0 is an eigenvalue of A, but is, for no φ, an eigenvalue of the linear L; then the number of eigen elements of A corresponding to λ_0 is finite. *M. Golomb.*

Rutic'kiĭ, Ya. B. On a nonlinear operator acting on an Orlicz space. Dopovidi Akad. Nauk Ukrain. RSR 1952, 161–166 (1952). (Ukrainian. Russian summary)

Let G be a bounded measurable set of n-dimensional Euclidean space and $f(x, u)$ a function continuous in u for almost all $x \in G$ and measurable in x for all u $(-\infty<u<\infty)$. The author studies the continuity of the operator

$$\mathbf{f}\varphi(x)=f(x, \varphi(x)) \quad (x \in G)$$

as acting from an Orlicz space $L^*_{M_1}$ to the space L_{M_2} [for definitions and notations see review of paper by Krasnosel'skiĭ and the author, Doklady Akad. Nauk SSSR (N.S.) 81, 497–500 (1951); these Rev. 13, 357]. He proves results analogous to those proved by Krasnosel'skiĭ [ibid. 77, 185–188 (1951); these Rev. 12, 836] for operators $\mathbf{f}$ acting on an L^p space. Thus, the first theorem states that if $\mathbf{f}$ takes $L^*_{M_1}$ into L_{M_2} then $\mathbf{f}$ is bounded in every sphere of $L^*_{M_1}$. *M. Golomb* (Lafayette, Ind.).

Ganapathy Iyer, V. A note on the linear space generated by a sequence of integral functions. J. Indian Math. Soc. (N.S.) 17 (1953), 183–185 (1954).

If $\alpha=\sum_{k=0}^{\infty}a_kz^k$ is an integral function, let $(\alpha)_n=\sum_{k=0}^{\infty}a_k{}^nz^k$. The main contribution of this note is a proof that if all of the a_k are distinct and non-zero, then every integral function can be obtained as a uniform limit, in any finite circle, of elements of the linear space generated by the $(\alpha)_n$, $n=1, 2, \cdots$. This generalizes an earlier result of the author [Proc. Amer. Math. Soc. 3, 874–883 (1952), p. 876; these Rev. 14, 657]. *M. Henriksen* (Lafayette, Ind.).

Morgenstern, Dietrich. Verschärfung eines Vollständigkeitskriteriums von Kaczmarz und Steinhaus. Math. Nachr. 11, 191–192 (1954).

$\{x^n\}$ $(n=0, 1, 2, \cdots)$ is complete in $L_w{}^p(-\infty, \infty)$ (norm: $\|f\|^p=\int_{-\infty}^{\infty}w(x)|f(x)|^pdx$), if $\limsup \|x^n\|^{1/n}/n<\infty$. The method of proof is the same as in E. Hewitt, Amer. Math. Monthly 61, 249–250 (1954); these Rev. 15, 631.

W. H. J. Fuchs (Ithaca, N. Y.).

***Cooke, Richard G. Linear operators. Spectral theory and some other applications.** Macmillan and Co., Ltd., London, 1953. xii+454 pp. $10.00.

Chapter 1 deals with some basic properties of the space L_2 and defines abstract Hilbert space (separability is assumed). (Well known theorems, as the Cauchy-Schwarz inequality (p. 25) and the triangle inequality (p. 26) for positive definite Hermitian bilinear forms are attributed to a colleague of the author, H. S. Allen.) Chapter 2 sketches the quantum theories of Heisenberg and Schrödinger. Though two of Rellich's papers on perturbation theory are inserted in the bibliography, there is no mention in the text of the existence of rigorous treatments of the subject. Chapter 3 deals with some basic properties of operators including the deficiency indices of Hermitian operators. (The theorem on the existence of an element $\neq 0$, orthogonal to a non-dense linear manifold, is credited to the same H. S. Allen (p. 99).) Chapters 4 and 5 (some 130 pages) reproduce some of the proofs of the spectral decomposition of self-adjoint operators (those of von Neumann, Lengyel, Lengyel-Stone, Cooper, Riesz-Lorch, etc.). It is hard to see the practical use of such lengthy reproductions of proofs, some of which seem now rather out of date (e.g., which use essentially the separability of the space). Chapter 6 is quite independent of the rest of the book: it deals with work of Köthe and Toeplitz, Weber, and Allen, on "matrix spaces and rings," and presupposes in terminology and notation a previous book of the author [Infinite matrices and sequence spaces, Macmillan, London, 1950; these Rev. 12, 694]. Chapter 7 introduces Gelfand's theory of normed rings, with the applications to Wiener's theory of absolutely convergent Fourier expansions.

Most chapters include some examples, and a list of works to be consulted, some of which, however, have almost no connection with the text and serve seemingly only to increase the bibliography placed at the end of the book (where Henri Cartan is identified with Elie Cartan). There is also an index of names and a general index (the former including more references to the name Cooke than to von Neumann and Stone together). There are some oddities in the notations: the space L_2 is denoted by $\sigma_2(f)$, and the 0-element of a linear space by z. This book may hardly be considered as a real gain in the growing literature of textbooks on operators. *B. Sz.-Nagy* (Szeged).

Sz.-Nagy, Béla. Transformations de l'espace de Hilbert, fonctions de type positif sur un groupe. Acta Sci. Math. Szeged 15, 104–114 (1954).

Let $\{T_\gamma\}$ be a family of linear bounded operators on a real or complex Hilbert space H where T_γ depends upon a variable element γ of a group Γ. In particular, if ϵ is the unit element of Γ then $T_\epsilon=I$. Suppose that T_γ is a function

of positive type on Γ in the sense that

$$T_{\gamma^{-1}}=T_\gamma^*,\quad \sum_{\gamma,\delta\,\epsilon\,\Gamma}=(T_{\delta^{-1}\gamma}g_\gamma,g_\delta)\geq 0$$

for every set of elements g_γ in H such that $g_\gamma=0$ except for a finite number of γ's. Suppose also that T_γ is weakly continuous in γ. Then there exists a Hilbert space $\mathbf{H}$ of which H is a subspace and a family of unitary operators U_γ in $\mathbf{H}$, forming a strongly continuous representation of Γ, such that $T_\gamma=PU_\gamma$ where P is the orthogonal projection of $\mathbf{H}$ onto H. Then $\mathbf{H}$ and U_γ are determined up to an isomorphism by the requirement that $U_\gamma f$ shall span $\mathbf{H}$ for $f\,\epsilon\,H$, $\gamma\,\epsilon\,\Gamma$. In particular, if T is a contraction in H and $T_n=T^n$ for $n\geq 0$ and $T_n=(T^*)^{|n|}$ for $n\leq -1$, then T_n is of positive type on the additive group of integers. Similarly, if $\{T_t|0\leq t<\infty\}$ is a weakly continuous semi-group of contractions in H and $T_t=T_{-t}^*$ for $t<0$, then T_t is of positive type on the additive group of reals. The conclusion concerning the representation of T_γ in these two special cases had been proved by the author by other methods in a recent paper [same Acta 15, 87–92 (1953); these Rev. 15, 326]. He also obtains a theorem of M. A. Neumark as a special case (a family $\{F_\lambda|-\infty<\lambda<+\infty\}$ of bounded selfadjoint transformations such that $F_\lambda\leq F_\mu$ for $\lambda<\mu$, $F_{\lambda+0}=F_\lambda$, $F_\lambda\to 0$ when $\lambda\to-\infty$ and to I when $\lambda\to+\infty$ may be represented as $F_\lambda=PE_\lambda$ where E_λ is a spectral family of projections in a space $\mathbf{H}$ containing H as a subspace and P is the projection on H). The first special case admits of a partial extension to an arbitrary real or complex Banach space: If T is a contraction in a (B)-space B there exists another (B)-space $\mathbf{B}$ admitting B as a subspace and an isometric transformation U of $\mathbf{B}$ onto B such that $T^nf=PU^nf$ for every $f\,\epsilon\,B$, $n=0,1,2,\cdots$, where P is the parallel projection of $\mathbf{B}$ onto B of norm one. *E. Hille* (New Haven, Conn.).

Krasnosel's'kiĭ, M. O. Approximate computation of characteristic values and functions of perturbed operators. Dopovidi Akad. Nauk Ukrain. RSR **1952**, 155–160 (1952). (Ukrainian. Russian summary)

Suppose A is a completely continuous operator in a real Hilbert space H, $Ae_0=\lambda_0e_0$, $\lambda_0>0$, $\|e_0\|=1$. The operator B defined by $B\varphi=\varphi-(\varphi,e_0)e_0-\lambda_0^{-1}A\varphi$ has a bounded inverse. Every eigenvector of the perturbed operator $K=A+D$, where D is a linear operator of sufficiently small bound, can be found as a solution of the nonlinear equation (a) $\|\psi\|K\psi-\psi=0$, the corresponding eigenvalue being $1/\|\psi\|$. The author shows that the sequence

$$\psi_0=\lambda_0^{-1}e_0,\quad \psi_n=\psi_{n-1}+B^{-1}(\|\psi_{n-1}\|K\psi_{n-1}-\psi_{n-1})$$

converges to a solution of (a) provided $\|D\|<\delta$ where δ is a certain function of λ_0, $\|B^{-1}\|$, $\|B^{-1}A\|$. The advantages of this seemingly laborious procedure over others of established use are not discussed. *M. Golomb* (Lafayette, Ind.).

Kreĭn, M. G. On the trace formula in perturbation theory. Mat. Sbornik N.S. **33** (75), 597–626 (1953). (Russian)

In this paper the author presents a rigorous mathematical study of a formula heuristically derived by Livšic [Uspehi Matem. Nauk (N.S.) 7, no. 1(47), 171–180 (1952); these Rev. 14, 185]. He considers a self-adjoint operator H on a Hilbert space, and a self-adjoint perturbing operator T; given a real function $\Phi(\lambda)$ on the real axis, he derives the trace formula

$$S\{\Phi(H+T)-\Phi(H)\}=\int_{-\infty}^{\infty}\xi(\lambda)d\Phi(\lambda) \tag{1}$$

for the trace of operator $\Phi(H+T)-\Phi(H)$ under various hypotheses. The function $\xi(\lambda)$ is independent of Φ and can be calculated from the operators H and T. The trace $S\{A\}$ of an operator A is defined for A in the trace class, the set of completely continuous operators A for which the series of eigenvalues, observing multiplicities, converges absolutely. The trace $S\{A\}$ is defined as this sum.

The author gives first a detailed treatment of the function

$$S(z)=S\{(H+T-zI)^{-1}-(H-zI)^{-1}\}$$

which corresponds to $\Phi(\lambda)=(\lambda-z)^{-1}$. When T is in the trace class he shows that $(H+T-zI)^{-1}-(H-zI)^{-1}$ belongs to the trace class for all non-real z. He next proves, in three steps, that formula (1) holds in this case. First, the case where T has a one-dimensional range, for some vector φ and real scalar τ: $T\psi=\tau(\psi,\varphi)\varphi$ for all vectors ψ; in this case we have the explicit formula

$$\xi(\lambda)=\frac{1}{\pi}\lim_{y\to 0}\arg\Delta(\lambda+iy) \tag{2}$$

for the function $\xi(\lambda)$, where $\Delta(z)=1+\tau((H-zI)^{-1}\varphi,\varphi)$. Second, if T has a finite-dimensional range it may be considered as a succession of one-dimensional perturbations and formula (1) carries over. Third, if T belongs to the trace class it can be approximated by finite-dimensional operators T_i and the corresponding ξ_i converge in $L^1(-\infty,\infty)$ to a ξ for which (1) holds. Using this basic result the author establishes (1) for various classes of functions $\Phi(\lambda)$, although he cannot characterize the set of admissible functions. Next he establishes a generalization of the trace function $S\{\Phi(H+T)-\Phi(H)\}$ for which (1) holds although the trace cannot exist in the usual sense.

Finally, he discusses another formula of Livšic; taking $\Phi_\mu(\lambda)$ as 1 for $\lambda\leq\mu$ and zero elsewhere and substituting formally in (1) gives

$$\xi(\mu)=-S\{\tilde{E}_\mu-E_\mu\} \tag{3}$$

where E_μ is the spectral decomposition of H and $\tilde{E}$ that of $H+T$. The author observes that (3) holds only in special cases and gives an example where $\tilde{E}_\mu-E_\mu$ does not belong to the trace class, so that $S\{\tilde{E}_\mu-E_\mu\}$ is not defined, and calculates $\xi(\mu)$ for this example. *D. C. Kleinecke.*

Jamison, S. L. Perturbation of normal operators. Proc. Amer. Math. Soc. **5**, 103–110 (1954).

Let $N(\epsilon)=N_0+\epsilon N_1+\epsilon^2N_2+\cdots$ be a bounded normal operator in Hilbert space $\mathfrak{H}$, depending regularly on the complex parameter ϵ for $|\epsilon|<\rho$. Let λ_0 be an isolated point of the spectrum of N_0: an eigenvalue of finite multiplicity m. Then there exist m numbers $\lambda^{(k)}(\epsilon)$ and m elements $\phi^{(k)}(\epsilon)$ of $\mathfrak{H}$ $(k=1,\cdots,m)$, depending regularly on ϵ, such that

$$N(\epsilon)\phi^{(k)}(\epsilon)=\lambda^{(k)}(\epsilon)\phi^{(k)}(\epsilon),\quad (\phi^{(k)}(\epsilon),\phi^{(l)}(\epsilon))=\delta_{kl},$$

and that the spectrum of $N(\epsilon)$ in a neighborhood of λ_0 consists, for sufficiently small $|\epsilon|$, of the eigenvalues $\lambda^{(k)}(\epsilon)$, each counted according to multiplicity. This is a generalization of Rellich's theorem on the perturbation of self-adjoint operators [Math. Ann. 113, 600–619 (1936)]; the technique used is partially that of B. Sz.-Nagy [Comment. Math. Helv. 19, 347–366 (1947); Acta Sci. Math. Szeged 14, 125–137 (1951); these Rev. 8, 589; 13, 849]. The author's method is to reduce the case of normal operators to the case of self-adjoint operators. [For other proofs of the same theorem, see F. Wolf, Math. Ann. 124, 317–333 (1952); these Rev. 14, 288; and T. Katô, J. Math. Soc. Japan 4, 323–337 (1952); these Rev. 14, 990.] *B. Sz.-Nagy.*

Guy, Roland. Sur une équation vectorielle intégrale dans un espace de Hilbert abstrait. C. R. Acad. Sci. Paris **238**, 46–49 (1954).

The author considers linear integral equations of Volterra type of the form

$$x(t)=x^0(t)+\int_{t_0}^{t}\mathfrak{F}(t,\tau)x(\tau)d\tau,$$

where $x(t)$ and $x^0(t)$ take values in Hilbert space, and $\mathfrak{F}(t,\tau)$ has self-adjoint operators (possibly unbounded) as values. Solutions are obtained by a process of successive approximation in which the spectral resolution of the operators $\mathfrak{F}(t,\tau)$ is used. *F. Smithies.*

Kadison, Richard V. Infinite general linear groups. Trans. Amer. Math. Soc. **76**, 66–91 (1954).

Let M be any Murray-von Neumann factor and let M_θ be the group of all invertible operators in M. These are the "general linear groups" considered here. This paper is a continuation of a study begun by the author in a previous paper [same Trans. **72**, 386–399 (1952); these Rev. **14**, 16] where the "infinite unitary group" M_u of unitary operators in M was considered. The following main results are obtained here. (1) If M is any factor and G is a (uniformly) closed normal subgroup of M_θ not contained in the center of M, then G contains a noncentral unitary operator, and if M is of type II_1, or III, then G contains M_u. (2) If M is of type III, then M_θ contains no proper, closed, noncentral, normal subgroups. (3) If M is any finite factor, then every closed, noncentral, normal subgroup of M_θ is the inverse image under the determinant function Δ [cf. Fuglede and Kadison, Ann. of Math. (2) **55**, 520–530 (1952); these Rev. **14**, 660] of a closed subgroup of the range of Δ. Moreover, the subgroup of M_θ consisting of those operators with determinant 1 has as its closed normal subgroups the closed central (scalar) subgroups. (4) Let $M_{\theta f}$ be the closure of the set of those operators in M_θ equal to a scalar multiple of the identity in the orthogonal complement of a subspace with finite relative dimension. Then $M_{\theta f}$ is a closed normal subgroup of M_θ consisting of those operators in M_θ the constituents of whose polar decompositions have at most one "center of infinite density" (cf. the author's paper on unitary groups cited above). If M is finite, $M_{\theta f}=M_\theta$; if M is of type III, $M_{\theta f}=\{\lambda I\}$; if M is of type I_∞ or II_∞, $M_{\theta f}$ is proper and non-central. (5) If M is of type I_∞ and G is a closed, noncentral, normal subgroup of M_θ, then G is the direct product of a closed subgroup of the scalars and the group $M_{\theta f(1)}$ equal to the closure of the set of those operators in M_θ which act as the identity of the complement of a subspace of finite relative dimension. *C. E. Rickart.*

Iséki, Kiyoshi. Sur les anneaux normés de Hilbert. II. Sur un théorème de M. W. Ambrose. C. R. Acad. Sci. Paris **237**, 545–546 (1953).

The author continues his discussion [same C. R. **236**, 1123–1125 (1953); these Rev. **14**, 883] of rings H which are also Hilbert spaces. He assumes an identity element and observes that H is a direct sum of matrix algebras [Ambrose, Trans. Amer. Math. Soc. **57**, 364–386 (1945); these Rev. **7**, 126]. *C. E. Rickart* (New Haven, Conn.).

Dixmier, Jacques. Sur les anneaux d'opérateurs dans les espaces hilbertiens. C. R. Acad. Sci. Paris **238**, 439–441 (1954).

Let $\mathfrak{A}$ be a ring of operators on a Hilbert space H. If E' is a projection in $\mathfrak{A}'$, then the mapping $T\rightarrow E'T$ of $\mathfrak{A}$ (onto a ring of operators on the range of E') is called an "induction" of $\mathfrak{A}$. If $\bar{H}$ is a Hilbert space direct sum of copies of H, then the natural mapping which extends each operator on H to an operator on $\bar{H}$ is called an "ampliation" of $\mathfrak{A}$. For $x\in H$, write $\omega_x(T)=(Tx,x)$, $T\in\mathfrak{A}$. The following are some of the results outlined here. 1. Let φ be a positive linear functional on $\mathfrak{A}$. If φ is normal, then there exists a sequence $\{x_i\}\subset H$ such that $\sum\|x_i\|^2<\infty$ and $\varphi=\sum\omega_{x_i}$. [This improves a result obtained by the author in Bull. Soc. Math. France **81**, 9–39 (1953); these Rev. **15**, 539.] If in addition $\mathfrak{A}$ has a separating vector, then φ is of the form ω_x. [This generalizes a result of H. A. Dye, Trans. Amer. Math. Soc. **72**, 243–280 (1952); these Rev. **13**, 662.] 2. Every normal homomorphism of $\mathfrak{A}$ onto a second ring of operators $\mathfrak{B}$ is a product of an "ampliation," an "induction" and a spatial isomorphism. 3. If the vector x (resp. y) is both generating and separating for $\mathfrak{A}$ (resp. $\mathfrak{B}$), then every isomorphism of $\mathfrak{A}$ on $\mathfrak{B}$ is spatial [cf. Griffin, ibid. **75**, 471–504 (1953); these Rev. **15**, 539]. 4. If $\mathfrak{A}$ is semi-finite, then it is "standard" if, and only if, it possesses a vector which is both generating and separating [cf. Segal, Ann. of Math. (2) **57**, 401–457 (1953); these Rev. **14**, 991]. *C. E. Rickart.*

Pallu de La Barrière, R. Algèbres unitaires et espaces d'Ambrose. Ann. Sci. Ecole Norm. Sup. (3) **70**, 381–401 (1953).

An Ambrose space is a Hilbert space with a partially defined multiplication and a notion of adjoint which satisfy certain axioms equivalent (but not obviously so) to the axioms given by Ambrose in defining the notion of H system. In part I of this paper the author establishes the equivalence, discusses in detail the close connection between the notion of Ambrose space and that of unitary algebra and gives a proof of the known theorem to the effect that the weakly closed algebras generated by left and right multiplication are commutators of one another. In part II he gives a proof of the fact that the Hilbert space of square summable functions on a unimodular locally compact group is an Ambrose space. Part III contains a definition of tensor product for Ambrose spaces and a proof of the fact that the Ambrose space of the direct product of two groups is the tensor product of their Ambrose spaces. Part IV deals with the center of an Ambrose space and to a limited extent with the classification of Ambrose spaces. Typical results: The center of an Ambrose space is a commutative Ambrose space. A necessary and sufficient condition that an irreducible Ambrose space have an identity element is that it be of finite class. Some of the results of the paper were announced and one was proved in an earlier note [C. R. Acad. Sci. Paris **233**, 997–999 (1951); these Rev. **13**, 473]. *G. W. Mackey* (Cambridge, Mass.).

Cristescu, Romulus. L'intégration dans les espaces semi-ordonnés. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 291–310 (1952). (Romanian. Russian and French summaries)

This article extends Bochner's integral for vector-valued functions to functions with values in a partially ordered linear space and to measures whose values are also in a partially ordered linear space. The usual theorems are obtained. *E. Hewitt* (Seattle, Wash.).

Theory of Probability

Moy, Shu-Teh Chen. Characterizations of conditional expectation as a transformation on function spaces. Pacific J. Math. 4, 47–63 (1954).

(a) Etant donnés un espace probabilisé (Ω, F, μ), une variable aléatoire x sur cet espace (pouvant prendre les valeurs $\pm\infty$ avec probabilité positive), une σ-algèbre F_1 sur Ω telle que $F_1 \subset F$, l'auteur définit l'espérance mathématique $E\{x\}$ et, à l'aide du théorème de Radon-Nikodým, l'espérance mathématique conditionnelle $E\{x|F_1\}$ dont il indique les propriétés. (b) Soit S l'espace des fonctions $\geqq 0$ mesurables sur (Ω, F), T une application linéaire de S dans lui-même, telle que: $T(x+y)=Tx+Ty$; $T(\alpha x)=\alpha Tx$ (α nombre $\geqq 0$); si x est borné, Tx est borné; $T(xTy)=(Tx)\cdot(Ty)$; si la suite non-decroissante $x_n \to x$, $Tx_n \to Tx$. L'auteur montre que T est nécessairement de la forme: $Tx = E\{xg|F_T\}$, où F_T est une σ-algèbre $\subset F$, et g une fonction $\geqq 0$ mesurable, telle que $E\{g|F_T\}$ soit bornée. (c) L'auteur indique une extension où on substitue à S un espace L_p, T étant alors une opération linéaire dans L_p. *R. Fortet* (Paris).

Blum, Julius R. Two theorems on almost sure convergence. Proc. Amer. Math. Soc. 5, 253–255 (1954).

Soit $\{X_n\}$ une suite de variables aléatoires, σ_n^2 la variance de $Y_n = X_n - X_{n-1}$; a) si $\sum_n \sigma_n^2 < +\infty$ et si pour tout $\epsilon > 0$ il existe un entier $\geqq 0$ aléatoire $N(\epsilon)$ tel que $n \geqq N(\epsilon)$ et $|X_n| > \epsilon$ entrainent $X_n E[Y_{n+1}|Y_1, \cdots, Y_n] \leqq 0$ presque-sûrement, une condition nécessaire et suffisante pour que X_n converge presque sûrement est que $E[Y_{n+1}|Y_1, \cdots, Y_n]$ tende vers 0 presque-sûrement; b) si $\sup_n E(|X_n|) < +\infty$ et si $\sum_n E\{[E\{Y_{n+1}|X_1, \cdots, X_n\}]^+\} < +\infty$, X_n converge presque-sûrement (on a posé; $a^+ = \frac{1}{2}[a+|a|]$). *R. Fortet.*

Franckx, E. Généralisation d'un théorème de Borel. Trabajos Estadistica 4, 369–371 (1953). (Spanish summary)

It is noted that in the divergence part of the Borel-Cantelli lemma the assumption of independence may be replaced by an inequality. *K. L. Chung.*

Gartštein, B. N. On the limiting distribution of the extreme and mixed ranges of a variational series. Dopovidi Akad. Nauk Ukrain. RSR 1951, 25–30 (1951). (Ukrainian. Russian summary)

Let $x_1, \cdots, x_n$ be independent variables with the same distribution $F(x)$, and let $\xi_1^{(n)} \leqq \xi_2^{(n)} \leqq \cdots \leqq \xi_n^{(n)}$ be their rearrangement in increasing order. The sample range of order (r, k) is defined as $\rho_{rk}^{(n)} = \xi_k^{(n)} - \xi_r^{(n)}$. The author gives an enumeration of the possible limiting distributions of these random variables as $n \to \infty$ and $n-k$ and r are either fixed, or $rn^{-1} \to \lambda$, or $kn^{-1} \to \lambda$. *W. Feller* (Princeton, N. J.).

*†***Cox, D. R., and Smith, Walter L. A direct proof of a fundamental theorem of renewal theory.** Skand. Aktuarietidskr. 36, 139–150 (1953).

The proof assumes the finiteness of the second moment and an order condition of the characteristic function at infinity, both superfluous for the fundamental theorem. Some generalizations are given. *K. L. Chung.*

Smith, Walter L. Asymptotic renewal theorems. Proc. Roy. Soc. Edinburgh. Sect. A. 64, 9–48 (1954).

A sequence of independent, non-negative and identically distributed random variables $\{X_i\}$ is called a renewal process, and if the X's may take only values which are multiples of a positive number it is called discrete. Let $N(x)$ be the greatest k such that $X_1 + \cdots + X_k \leqq x$ (>0), $H(x) = E(N(x))$. If $K(x)$ is the distribution function of any non-negative random variable with mean $\kappa > 0$, then for a non-discrete process

$$\lim_{x\to\infty} \left\{ H(x) - \int_0^x K(x-s)dH(s) \right\} = \kappa/E(X_i);$$

a similar result (after smoothing) for a discrete process. The proof uses Pitt's form of Wiener's general Tauberian theorem. A number of results are derived from this general theorem including Blackwell's theorem [Duke Math. J. 15, 145–150 (1948); these Rev. 9, 452], extension of several of Feller's theorems [Trans. Amer. Math. Soc. 67, 98–119 (1949); these Rev. 11, 255] to the non-discrete process, and an improvement of Doob's theorem on limiting age distribution [ibid. 63, 422–438 (1948); these Rev. 9, 598]. Conditions are also established under which $\lim_{x\to\infty} H'(x) = 1/E(X_i)$. *K. L. Chung* (Syracuse, N. Y.).

Gihman, I. I. On some limit theorems for conditional distributions and on problems of mathematical statistics connected with them. Ukrain. Mat. Žurnal 5, 413–433 (1953). (Russian)

Let $x_1, x_2, \cdots$ be mutually independent random variables with a common distribution having mean 0 and variance 1. It is supposed that the common distribution is either of lattice type or is determined by a density function of bounded variation. Let $n^{1/2}x_{nk} = \sum_1^k x_j$, let z_n be a possible value of x_{nn}, and let $z_n \to z$. Then certain limiting $(n\to\infty)$ conditional distributions are found, for $x_{nn} = z_n$. The evaluation is obtained in each case as a solution of a parabolic differential equation with appropriate boundary conditions. (1) Let φ, ψ be continuously differentiable functions on $(0, 1)$, with $\varphi(0) < 0 < \psi(0)$, $\varphi(t) < \psi(t)$. Then the limiting probability that $\varphi(k/n) < x_{nk} < \psi(k/n)$, $k \leqq Tn$, is found. (2) The limiting conditional distributions $\max_{k\leqq n} x_{nk}$ and $\max_{k\leqq nt} [x_{nk} - kz_n/n]$ are found. (3) The limiting conditional distribution of the number divided by n of positive terms in $x_{nk} - kz_n/n$, $k \leqq n$, is uniform on $(0, 1)$.

In a second section the author finds the limiting distribution $(n\to\infty)$ of the difference between empirical and true distributions of a sample of n (here treated by grouping the observations and making the grouping finer as $n\to\infty$) when the unknown distribution depends on a parameter and the parameter is estimated in various ways. The results are too complicated to summarize here. The mathematical basis for the author's important results, particular cases of some of which have been obtained previously, is contained in a previous (unavailable) paper [Mat. Sbornik Kiev. Univ. 8]. *J. L. Doob* (Urbana, Ill.).

Derman, C. A solution to a set of fundamental equations in Markov chains. Proc. Amer. Math. Soc. 5, 332–334 (1954).

Soit une chaine de Markoff discrète, à une infinité dénombrable d'états possibles E_i $(i=0, 1, 2, \cdots)$, homogène, irréductible, de matrice de transition $P(p_{ij})$, $p^n{}_{ij}$ désignant l'élément de P_j^n; posons: $\Pi_i^n = \sum_{r=0}^n p^r{}_{ii}$, quantité qui s'interprète aisément; K. L. Chung [J. Research Nat. Bur. Standards 50, 203–208 (1953); ces Rev. 14, 1099] a montré que, si E_i et E_j sont récurrents, $\lim_{n\to+\infty} \Pi_j^n/\Pi_i^n$ existe. Si tous les E_i sont récurrents, l'auteur en déduit qu'il existe un système et un seul de nombres v_k tels que (a) $v_0 = 1$; (b) $v_j > 0$ $(j=1, 2, \cdots)$; (c) $v_j = \sum_{i=0}^\infty v_i p_{ij}$, à savoir les nombres $v_k = \lim_{n\to+\infty} \Pi_k^n/\Pi_0^n$. *R. Fortet* (Paris).

†The stated theorem actually deals with the convergence of renewal densities for which better conditions are given in the paper by Smith reviewed following this one.

Hittmair, Otto. Valeur extrémale des distributions de probabilités conditionnelles dans une chaîne de Markoff. C. R. Acad. Sci. Paris **238**, 1469–1470 (1954).

Let $x(t)$ be a Markov process. The conditional probability of a value $x(t+\tau)$ given the value $x(t)$ at time $t+\tau$ expressed through its values at p intermediate time points is given by

$$f(x(t+\tau)|x(t))=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(x_{p+1}|x_p),\cdots,\\ f(x_1|x_0)dx_p\cdots dx_1,$$

where $x_{p+1}=x(t_{p+1})=x(t+\tau)$ and $x_0=x(t)$. Let

$$\varphi(x_i)=f(x_{i+1}|x_i)f(x_i|x_{i-1}),\quad\text{and}\quad m_i=\int_{-\infty}^{\infty}x_i\varphi(x_i)dx_i.$$

The author remarks that if $f(x_{i+1}|x_{i-1})/\varphi(m_i)$ is constant independent of x_{i+1} and x_{i-1} then

$$f(x(t+\tau)|x(t))=c_pf(x_{p+1}|m_p)f(m_p|m_{p-1})\cdots f(m_1|x_0),$$

and if $d\varphi(x_i)/dx_i|_{x_i=m_i}=0$, then

$$f(x(t+\tau)|x(t))=c_p\max_{x_1,\cdots,x_p}f(x_{p+1}|x_p)\cdots f(x_1|x_0).$$

In particular this will be true if $f(x_{i+1}|x_i)$ is Gaussian.

J. L. Snell (Princeton, N. J.).

Hittmair, Otto. Principe extrémal d'une chaîne de Markoff dans le mouvement brownien. C. R. Acad. Sci. Paris **238**, 1555–1557 (1954).

The author points out the connection of the result of the paper reviewed above with certain well known results on the velocity and displacement of a free particle undergoing Brownian motion. *J. L. Snell* (Princeton, N. J.).

Ramakrishnan, Alladi, and Mathews, P. M. On a class of stochastic integro-differential equations. Proc. Indian Acad. Sci. Sect. A. **38**, 450–466 (1953).

Two stochastic processes of the Markov type are studied. I. A particle looses or gains energy at a constant rate β. In addition, the particle occasionally looses instantaneously a fixed part γE of its energy E. The times of such jumps constitute a Poisson process independent of the energy. An expression for the probability distribution of the energy distribution has been given by Chandrasekhar and Münch [Astrophys. J. **112**, 380–392, 393–398 (1950); these Rev. **12**, 644]. The author's principal object in this part of his study seems to be to point out that when the rate β is negative the process stops when the energy is reduced to zero. The probability of this occurring before t is called a delta function singularity.

II. The second model is a variation of the first: one puts $\beta=0$ and assumes that at the jumps the energy E changes to E' so that E'/E is a uniformly distributed random variable in $(0,1)$. The process stops when an energy threshold E_0 is reached. While only these two cases are studied the author also discusses in general the integro-differential equation of what the reviewer called the mixed process, i.e., diffusion plus jumps; however he puts the diffusion coefficient equal to zero so that the continuous part of the motion is deterministic. [For definition and solution of the equations cf. Feller, Math. Ann. **113**, 113–160 (1936).]

W. Feller (Princeton, N. J.).

Ramakrishnan, Alladi, and Mathews, P. M. A stochastic problem relating to counters. Philos. Mag. (7) **44**, 1122–1128 (1953).

The usual model of a counter with a dead time after each event occurring is generalized by introducing different dead times following registered and unregistered events.

W. Feller (Princeton, N. J.).

Davies, R. W. The connection between the Smoluchowski equation and the Kramers-Chandrasekhar equation. Physical Rev. (2) **93**, 1169–1170 (1954).

It is known that the probability density, $W(x,v;t)$, in phase space, of a particle describing Brownian motion is governed by the differential equation

$$W_t+vW_x=\{[\beta v-K(x)]W\}_v+(kT\beta/m)W_{vv},$$

where $K(x)$ is an external force, T is the temperature, m is the mass and k is Boltzmann's constant. From this equation one readily obtains by integration

$$\rho_t+(\rho\bar{v})_x=0, \tag{1}$$

and

$$(\rho\bar{v})_t+(\rho\langle v^2\rangle)_x+\beta\rho\bar{v}=K(x)\rho, \tag{2}$$

where $\rho(x,t)=\int_{-\infty}^{+\infty}W(x,v;t)dv$ and $\rho\langle v^2\rangle=\int_{-\infty}^{+\infty}v^2Wdv$. Eliminating $\rho\bar{v}$ from the foregoing two equations we obtain

$$\frac{1}{\beta}\rho_{tt}+\rho_t+\left(\frac{K(x)\rho}{\beta}\right)_x=\left(\frac{\rho\langle v^2\rangle}{\beta}\right)_{xx}. \tag{3}$$

On the other hand, it has been stated in the literature that for times $t\gg\beta^{-1}$, the probability density ρ is governed by the Smoluchowski equation

$$\rho_t+\left(\frac{K(x)\rho}{\beta}\right)_x=\frac{kT}{m\beta}\rho_{xx}. \tag{4}$$

The author discusses the sense in which solutions of equation (4) may be considered as the limiting forms of the solutions of (3); and shows, in particular, that in order to obtain equation (4) in the limit $\beta\to\infty$, we must let $\langle v^2\rangle$ and $K(x)$ also approach infinity in such a way that

$$\lim_{\beta\to\infty}(\langle v^2\rangle/\beta)=kT/6\pi a\eta,\quad \lim_{\beta\to\infty}K(x)/\beta=U(x), \tag{5}$$

where $U(x)$ is the "drift" velocity, η is the absolute viscosity of the liquid surrounding the particles and a is the particle radius. *S. Chandrasekhar* (Williams Bay, Wis.).

Łopuszański, J. Lösung der G-Gleichungen von Jánossy für die kosmischen Schauer. Acta Phys. Polonica **12**, 156–159 (1953).

The so-called G-equations used by Jánossy [Proc. Phys. Soc. Sect. A. **63**, 241–249 (1950)] and by Jánossy and Messel [ibid. **63**, 1101–1115 (1950)] are solved by successive approximation. For a value of the occurring parameter ϵ the equations admit of a constant solution. From this one obtains successively solutions in the range $\epsilon>1/2$, $\epsilon>1/3$, etc.

W. Feller (Princeton, N. J.).

Jauho, Pekka. The number of artificial nuclear reactions as a random variable. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **165**, 12 pp. (1954).

Suppose that incident particles can hit (and destroy) target particles. The incident particles represent, by assumption, independent trials; at each, the probability of a hit is proportional to the number of target particles at that time. Let N_k be the number of trials required to obtain the k-k hit. Then the random variables N_k-N_{k-1} are mutually

independent and each has a geometric distribution. The generating function is therefore the reciprocal of a polynomial of degree k. The author obtains the probability distribution for N_k by complex variable methods. He also calculates the mean and variance [cf., e.g., Feller, An introduction to probability theory, Wiley, New York, 1950, pp. 175, 181; these Rev. 14, 1094]. *W. Feller.*

Lundkvist, Karl. General theory for telephone traffic. Ericsson Technics 9, 111–140 (1953).

The general theory mentioned in the title is an engineering version of the general birth and death process [W. Feller, An introduction to probability theory and its applications, v. 1, Wiley, New York, 1950; these Rev. 12, 424] and is recommended as a more adequate mathematical model for studies of telephone traffic, including the effects of graded multiple, being conducted with a traffic machine. The system states are regarded under the fourfold possibilities coming from beginning or ending a state with a birth or death, and it is shown that for stationary conditions the number of states beginning and ending with a birth is equal to the number of those beginning and ending with a death, for any population size. The stationary probabilities for traffic (population) size and traffic "loss" are written out for three cases. Finally three examples of the use of the theory are given in detail; these are: (i) random arrivals and exponential holding (service) times, (ii) the same but with constant holding times, a finite number of trunks and no waiting line, (iii) a single trunk for random arrivals and either limited or unlimited waiting lines served in order of arrival. It may be noted that for the last two the author follows Erlang in the questionable assumption that engaged trunks have been engaged at random in a holding time interval; however the results are said to be in agreement with known results.

J. Riordan (New York, N. Y.).

Mathematical Statistics

Have

***Kitagawa, Tosio. Tables of Poisson distribution.** Baifukan, Tokyo, 1952. 156 pp.

The book contains three tables of the Poisson probabilities $e^{-m}m^x/x!$. Table I covers the range

$$m = 0.001\,(0.001)\,1.000\,(8d),$$

Table II has the range $m = 1.01\,(0.01)\,5.00\,(8d)$ while Table III is for $m = 5.01\,(0.01)\,10.00\,(7d)$. Table IV gives 1% and 5% confidence limits for the mean of large samples by means of a procedure suggested by S. S. Wilks [Ann. Math. Statistics 9, 166–175 (1938)]. A discussion of single and double sampling inspection plans is given, this is an exposition of an earlier paper by the author [Seminar Reports of the Institute of Statistical Mathematics 2, 255–295 (1946) (in Japanese)]. The tables give more decimals and are tabulated, within their range, for finer gradations of m than Molina's tables [Poisson's exponential binomial limit, Van Nostrand, New York, 1942; these Rev. 4, 18]. However, Molina's table covers also the range $m = 10.0\,(0.1)\,15.0\,(1)\,100$. In addition, Molina gives also a table of the Poisson cumulative distribution function $P(c, m) = \sum_{x=c}^{\infty} m^x e^{-m}/x!$. Printing and appearance of the present table are excellent.

E. Lukacs (Washington, D. C.).

Linnik, Yu. V. Linear forms and statistical criteria. I, II. Ukrain. Mat. Žurnal 5, 207–243, 247–290 (1953). (Russian)

Let $x_1, \cdots, x_r$ be independent, identically distributed random variables with the common distribution function $F(x)$. Let $L_1(x) = \sum a_i x_i$, $L_2(x) = \sum b_i x_i$, where the a's and b's are real constants such that $\max|a_i| \neq \max|b_i|$. In order that the two statements (A) $F(x)$ belongs to the normal type, (B) the statistics $L_1(x)$ and $L_2(x)$ are identically distributed, be equivalent it is necessary and sufficient that (1) $\sum a_i = \sum b_i$; (2) $\sigma(2) = 0$ where $\sigma(z) = \sum |a_i|^z - \sum |b_i|^z$; (3) all positive roots of $\sigma(z)$ that are integers divisible by 4 are simple roots; (4) all positive roots of $\sigma(z)$ that are $\equiv 2 \pmod 4$ have multiplicity not exceeding 2; moreover, if there is such a double root, it is the unique maximum of all positive roots of $\sigma(z)$; (5) if $\sigma(z)$ has a positive root γ that is not an integer, then it is the unique maximum of all positive roots, it is simple, and $[\gamma/2]$ is odd. This is the main theorem [for other results see Doklady Akad. Nauk SSSR (N.S.) 89, 9–11 (1953); these Rev. 15, 42]. There seems however to be a gap in §52. In proving that $y_j(x) > 0$ the range $K_0 A^{1/2} \leq x \leq K_0 A^{1/2+\epsilon_0}$ is not covered. If this should be an error then Lemma XV and the theorem might have to be modified. *K. L. Chung* (Syracuse, N. Y.).

Illyašenko, O. A. On the influence of grouping of empirical data on A. N. Kolmogorov's criteria of fit. Dopovidi Akad. Nauk Ukrain. RSR 1952, 3–6 (1952). (Ukrainian. Russian summary)

The Kolmogorov formula for the limiting distribution of the difference between a sample distribution $F_n(x)$ and the theoretical distribution $F(x)$ holds only when F is continuous. The author derives analogous formulas for $F - F_n$ and $|F - F_n|$ when F is a multinomial distribution, i.e., when each variable can assume only s different values. The limiting distribution is given in the form of an $(s-1)$-tuple integral over an s-variate normal distribution.

W. Feller (Princeton, N. J.).

Gihman, I. I. On a criterion of fit for discrete random variables. Dopovidi Akad. Nauk Ukrain. RSR 1952, 7–9 (1952). (Ukrainian. Russian summary)

The author shows how the results of the preceding review can be used for testing the goodness of fit. *W. Feller.*

Gnedenko, B. V. Some remarks on the papers of O. A. Illyašenko and I. I. Gihman. Dopovidi Akad. Nauk Ukrain. RSR 1952, 10–12 (1952). (Ukrainian. Russian summary)

Moriguti, Sigeiti. A note on Hartley's formula of studentization. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 2, no. 4, 99–103 (1953).

The author gives a simplified proof of Hartley's [Biometrika 33, 173–180 (1944); these Rev. 6, 10] formula for the probability integral of a "studentised" integral: A statistic x with cumulative distribution function $F(x)$ is said to be proportional to the standard deviation σ of a normal parent if $F(x/\sigma)$ does not involve σ. The corresponding "studentised" statistic $t = x/s$ is then obtained with the help of an unbiassed estimate s^2 of σ^2, distributed independently of x as χ^2/n for n degrees of freedom. The cumulative distribution $F_n(t)$ of t can then be obtained in the form

$$(1) \qquad F_n(t) = F(t) + \sum_{i=1}^{\infty} b_i(t) n^{-i}$$

for which the coefficients $b_i(t)$ up to $i=2$ were obtained by Hartley [loc. cit.] and those for $i=3$, 4 by K. R. Nair [Biometrika 35, 16–31 (1948); these Rev. 9, 601]. These results are here reproved on the lines of Hartley's earlier paper [Suppl. J. Roy. Statist. Soc. 5, 80–88 (1938)] with the help of a Taylor expansion. This method provides upper bounds for the error in a truncated expansion (1).

H. O. Hartley (Ames, Iowa).

Weibull, Martin. The distributions of t- and F-statistics and of correlation and regression coefficients in stratified samples from normal populations with different means. Skand. Aktuarietidskr. 36, no. 1–2, supplement, 106 pp. (1953).

When the population sampled is not a homogeneous normal population, the question arises as to what the distributions of the common statistics t, F, etc. really are, and what inferences can be drawn from the common tests of significance. The author offers the alternative model that the observations are independent, normal, with common variance but means all different. The non-central chi-square and the doubly non-central t and F distributions are derived and discussed. Various properties of independence, limiting properties as degrees of freedom increase, etc. of the central statistics carry over to the non-central case. The effect of the non-centrality parameters on mean and variance is studied. A table gives $P(F>F_\alpha)$ for $\alpha=.01, .05$; $b_1=1, 2, 4, 8$; $b_1\leqq b_2=1, 2, 4, 8, 16, 32, \infty$; β_1^2 and $\beta_2^2=0, \frac{1}{2}, 1$; where b_1, b_2 are degrees of freedom, β_1^2, β_2^2 are non-centrality parameters for numerator and denominator, respectively. Particular statistics are then considered, such as standardized linear functions of several independent means, ratios in analysis of variance, etc. A final chapter derives the non-central Wishart distribution for a two-dimensional population, also the non-central distribution of the correlation coefficient.

S. W. Nash (Vancouver, B. C.).

Fraser, D. A. S. Completeness of order statistics. Canadian J. Math. 6, 42–45 (1954).

A set S of probability measures v on a Borel field of subsets of a space X is a complete class if $\int g dv=0$ for all $v \varepsilon S$ implies $g=0$ a.e. with respect to each v. Let u be a non-atomic measure over a Borel field $\mathfrak{B}$ of subsets of a space Y, let $\mathfrak{A}$ be a ring of subsets of Y which generates $\mathfrak{B}$, let T consist of all probability measures on $\mathfrak{B}$ whose density with respect to u is $c\phi$, where c is constant and ϕ is the characteristic function of a set in $\mathfrak{A}$, and let X be the n-fold product space of Y. Each measure in T determines a product measure v on the product Borel field $\mathfrak{C}$ of subsets of Y. The class S of these measures v, restricted to the Borel field $\mathfrak{D}$ of symmetric sets of $\mathfrak{C}$, is shown to be a complete class of measures.

D. Blackwell (Washington, D. C.).

Laha, R. G. On an extension of Geary's theorem. Biometrika 40, 228–229 (1953).

R. C. Geary [Suppl. J. Roy. Statist. Soc. 3, 178–184 (1936)] has shown that the independence of the sample mean and the sample variance implies the normality of the population, provided that moments of all orders exist. The reviewer [Ann. Math. Statistics 13, 91–93 (1942); these Rev. 4, 16] gave a different proof requiring only the existence of the second moment. Later T. Kawata and H. Sakamoto [J. Math. Soc. Japan 1, 111–115 (1949); these Rev. 11, 188] and independently A. A. Zinger [Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 172–175 (1951); these Rev. 13, 479] generalized the theorem by eliminating the assumption about the existence of moments of the population distribution. However, none of the proofs mentioned makes full use of the assumption of independence. The author generalizes Geary's theorem in a new direction. He maintains the assumption that the population has second moments but replaces the assumption of independence of the mean and the variance by a weaker statement involving conditional expectations.

E. Lukacs.

Hartley, H. O., and David, H. A. Universal bounds for mean range and extreme observation. Ann. Math. Statistics 25, 85–99 (1954).

Let x_1 and x_n be the smallest and largest of n independent observations on a chance variable x with c.d.f. F. The authors obtain that F which maximizes $E\{x_n\}$ subject to specified mean and variance of x. An upper bound for $E(x_m)$ is obtained where x_m is the mth order statistic. The best upper and lower bounds for $E(x_n-x_1)$ are obtained under the following restrictions. The mean and variance are 0 and 1 respectively and $a\leqq x\leqq b$ where a and b are given constants. The upper bounds are attained and the corresponding distributions derived. The lower bounds are attained by a discrete distribution where x may assume only two values. These results are of interest in industrial applications where there is considerable interest in the biases that may arise by using the range to estimate σ and assuming normality without justification.

H. Chernoff.

Gumbel, E. J. The maxima of the mean largest value and of the range. Ann. Math. Statistics 25, 76–84 (1954).

Let x_n be the largest of n independent observations on a continuous chance variable with c.d.f. F. The author derives the c.d.f. with specified mean and variance for which $E(x_n)$ is a maximum. He applies the same technique to derive the c.d.f. with specified variance for which the mean range is a maximum. The latter is a known result. The author uses the technique of the calculus of variations where the first variation is set equal to zero. He does not give the usual sufficiency arguments to show that the results so obtained furnish the solutions to his problem.

H. Chernoff.

de la Garza, A. Spacing of information in polynomial regression. Ann. Math. Statistics 25, 123–130 (1954).

Given N observed values of an error-free variable x and corresponding values of a random variable y. Assume there are p distinct values of x, designated as x_h, with n_h observations on y when $x=x_h$, $\sum_{h=1}^{p} n_h=N$. The relationship between y and x is given by $P(x_h)=y_{h\epsilon}+\delta_{h\epsilon}$, where

$$P(x_h)=\sum_{i=0}^{m}\alpha_i x_h^i;\quad h=1(1)p\geqq(m+1);\quad \epsilon=1(1)n_h.$$

The $\delta_{h\epsilon}$ are uncorrelated, with zero means and variances $1/w_{h\epsilon}$. The total information in the sample is given by $\sum w_{h\epsilon}$. If $p>(m+1)$, the author shows how to find a new set of $(m+1)$ x's, designated as r_j (n_j observations for each r_j), such that the total information is not changed. This problem has a practical interest, since it generally costs less to obtain a new observation for a given x than to use a different x. An application is given for $m=2$; the problem is to minimize the maximum variance of the least-squares estimate of $P(x)$ within the given range of x.

R. L. Anderson (Raleigh, N. C.).

Rippe, Dayle D. Application of a large sampling criterion to some sampling problems in factor analysis. Psychometrika 18, 191–205 (1953).

A technique is presented to test the completeness of factor solutions and also to test the significance of common-component loadings. The chi-square test involved is based on the asymptotic normality of the sample variances and covariances. Numerical examples illustrate the technique.
S. W. Nash (Vancouver, B. C.).

James, A. T. Normal multivariate analysis and the orthogonal group. Ann. Math. Statistics 25, 40–75 (1954).

The author suggests the use of certain new techniques, borrowed from differential and integral geometry, in obtaining sampling distributions based on a normal multivariate population, and gives alternative derivations of various known distributions by way of illustration. He devotes §§3–5 of the paper to expounding, in a form suitable for his applications, the theory of analytic manifolds, and of exterior differential forms and their integrals [cf. W. Blaschke, Integralgeometrie, Hermann, Paris, 1935; also S.-S. Chern, Sankhyā 7, 2–8 (1945); these Rev. 7, 194]. He deals especially with the group of orthogonal transformations of n-dimensional space R^n, the Grassmann manifold of k-planes (that is, k-dimensional linear subspaces) in R^n, and the Stiefel manifold of k-frames (that is, sets of k orthogonal unit vectors) in R^n. By treating these as analytic manifolds he avoids the loss of symmetry involved when particular (local) parametrizations are used, and by writing the element of measure (or probability) in a multiple integral as an exterior differential form (and using the anti-commutative multiplication appropriate to such forms) he avoids writing down bulky determinants arising as Jacobians in changes of variables. In §7, generalizing an approach due to Hotelling [Biometrika 28, 321–377 (1936)] he obtains an alternative derivation of the distribution of the canonical correlations for samples from normal populations in the null case: the result was found differently by R. A. Fisher [Ann. Eugenics 9, 238–249 (1939); these Rev. 1, 248] and others. In §8 he deals with an $n\times k$ matrix X, where $k\leqq n$, whose rows are n independent observations from a non-singular normal k-variate distribution with means zero. Writing $x_1, \cdots, x_k$ for the column-vectors constituting X, he shows (Theorem 8.1) that the distribution of a normal k-variate sample can be decomposed into three independent distributions: (a) essentially the Wishart distribution, (b) the invariant distribution of the k-plane spanned by the vectors $x_1, \cdots, x_k$, (c) the invariant distribution of the orthogonal $k\times k$ matrix that determines the orientation of $x_1, \cdots, x_k$ in this k-plane. (The invariance concerned is relative to transformations from X to HX, where H is any orthogonal $n\times n$ matrix.) He obtains incidentally a decomposition found previously by I. Olkin [Inst. Statist. Univ. of North Carolina. Mimeo. Ser. no. 43 (1951)] and thence the distribution of the latent roots of the sample variance-covariance matrix when the population latent roots are equal: this result was found otherwise by Fisher [loc. cit.]. The formulae involved in these various results are too elaborate for reproduction here.
H. P. Mulholland.

Kamat, A. R. Some properties of estimates for the standard deviation based on deviations from the mean and variate differences. J. Roy. Statist. Soc. Ser. B. 15, 233–240 (1953).

The author gives exact formulae for the relative variances of the following statistics, which are constructed from a sample of independent Gaussian variates: (1) the sum of the pth powers of the absolute deviations from the mean, $p=1, 2, 3, 4$; (2) the sum of the pth powers of absolute values of rth differences, $p, r=1, 2, 3, 4$. Corresponding asymptotic formulae are derived for the estimates of variance based on these statistics, and it is shown that the moments of the ratio of any such estimate to the usual sum-of-squares estimate are equal to the ratios of moments. The effect of a slow trend in the observations is evaluated approximately.
P. Whittle (Wellington).

Roy, S. N. and Bose, R. C. Simultaneous confidence interval estimation. Ann. Math. Statistics 24, 513–536 (1953).

Sufficient conditions are given for getting simultaneous confidence intervals for a finite or infinite set of parametric equations $\Pi_k=f_k(\theta)$. Suppose there is a set of functions $\psi_k(y, \Pi_k)$ such that $d_1\leqq\psi_k\leqq d_2$ with constants d_1, d_2 independent of k is equivalent to $\varphi_{k1}(y)\leqq\Pi_k\leqq\varphi_{k2}(y)$, where y is a set of random variables depending on θ, a set of parameters. Let $W_{k,\theta}=\{y: d_1\leqq\psi_k\leqq d_2\}$ and $W_\theta=\bigcap_k W_{k,\theta}$. If $P\{y\in W_\theta|\theta\}=1-\alpha$ is independent of θ, then $1-\alpha$ is also the chance that $\varphi_{k1}(y)\leqq\Pi_k\leqq\varphi_{k2}(y)$ is true for all k. Here W_θ is the set of points for which $d_1\leqq\inf_k\psi_k(y, \Pi_k)$ and $\sup_k\psi_k(y, \Pi_k)\leqq d_2$. The authors choose as their intervals $d_1\leqq\psi_k\leqq d_2$ ones having strong optimum properties. They then apply their theory to getting confidence intervals for linear combinations of population means from univariate and multivariate normal populations and for certain functions of the population covariance matrices and population canonical regressions in the latter case. The simple hypothesis H_0: $\Pi_k=\Pi_{k0}$ for all k can be tested by using as the region of rejection the complement of W_θ. This method of getting simultaneous confidence intervals has already been applied in analysis of variance by J. W. Tukey (unpublished) and H. Scheffé [Biometrika 40, 87–104 (1953); these Rev. 15, 239].
S. W. Nash (Vancouver, B. C.).

Freire, Rémy. L'estimation des paramètres des fonctions d'Engel. Publ. Inst. Statist. Univ. Paris 2, no. 3, 19–26 (1953).

The chance variables $Y_1, Y_2, \cdots, Y_n$ are independently and normally distributed with a common variance and with $EY_i=\alpha+\beta f(x_i, k_1, \cdots, k_m)$, where $x_1, x_2, \cdots, x_n$ are known constants, $f(x, k_1, k_2, \cdots, k_m)$ is a function of known form, and $\alpha, \beta, k_1, k_2, \cdots, k_m$ are unknown parameters. The author describes a method of computing approximately the maximum likelihood estimates of the parameters. It is essentially an iterative procedure, starting with tentative values of the estimates, but the author recommends one or at most two iterations. The discussion is not intended to be rigorous. Special attention is given to the particular case where $f(x, k_1, \cdots, k_m)=\log [x/\sum_1^m p_jk_j]$, $p_1, \cdots, p_m$ being known constants.
L. Weiss (Charlottesville, Va.).

Haldane, J. B. S. The estimation of two parameters from a sample. Sankhyā 12, 313–320 (1953).

The author states a method he proposed for estimating one parameter and applies it to estimating two parameters. The method, which is given without derivation or heuristic argument, is to find the pair of estimates x, y which minimize $\sum_{r=1}^m f_r^2(\xi, \eta)/(n_r+1)$ where ξ, η are the unknown parameters and f_r, n_r are respectively the known frequency function and the observed frequency of the rth of m given classes. The proposed method, called minimum discrepancy, is compared with the method of maximum likelihood. Like

the latter, the former gives consistent and efficient estimates; in fact $x-\xi$ and $y-\eta$ are shown to have the same leading term for both methods when expressed as a series in inverse powers of N, the total sample size. A specially chosen numerical example from genetics illustrates that the proposed method may lead to equations that are much easier to solve. *M. Sobel* (Allentown, Pa.).

Truax, Donald R. An optimum slippage test for the variances of K normal distributions. Ann. Math. Statistics 24, 669–674 (1953).

The problem is that of comparing the variances of k normal populations Π_i: $N(m_i, \sigma_i^2)$ with unknown means m_i and unknown variances σ_i^2 $(i=1, 2, \cdots, k)$, on the basis of n observations $x_{i1}, x_{i2}, \cdots, x_{in}$ from each Π_i. A population Π_i is said to have slipped to the right if $\sigma_1^2=\sigma_2^2=\cdots=\sigma^2_{i-1}=\sigma^2_{i+1}=\cdots=\sigma_k^2$ and $\sigma_i^2=\lambda^2\sigma_1^2$ where $|\lambda|>1$. Let D_0 be the decision that all k variances are equal, and let D_j be the decision that D_0 is false and $\sigma_j^2=\max(\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2)$ for $j=1, 2, \cdots, k$. A statistical procedure is found which will select one of the $k+1$ decisions $D_0, D_1, \cdots, D_k$, which, subject to the conditions: (a) when all variances are equal, D_0 should be selected with probability $1-\alpha$, α being assigned, (b) it is symmetric, that is, the probability of selecting D_i when $\sigma_1^2=\sigma_2^2=\cdots\sigma^2_{i-1}=\sigma^2_{i+1}=\cdots=\sigma_k^2$ and $\sigma_i^2=\lambda^2\sigma_1^2$ should be equal for all i, and (c)-(d) it is invariant if all the observations are subject to the same linear transformation, will maximize the probability of making the correct decision when one of the populations has slipped to the right. The solution is the following: if $s^2_M/\sum_{i=1}^k s_i^2>L_\alpha$, select D_M; if $s^2_M/\sum_{i=1}^k s_i^2<L_\alpha$, select D_0, where M denotes the population yielding the largest sample variance, L_α being a constant determined by condition (a). This statistic is coincident with that already suggested by Cochran. *T. Kitagawa.*

Matthai, Abraham. On selecting random numbers for large-scale sampling. Sankhyā 13, 257–260 (1954).

Good, I. J. The serial test for sampling numbers and other tests for randomness. Proc. Cambridge Philos. Soc. 49, 276–284 (1953).

In the serial test for randomness of sampling numbers a statistic ψ^2 is used. It is in the form of a sum of squares and was previously supposed to have asymptotically a chi-square distribution. The present paper shows that this is not so. Let ψ_ν^2 denote the statistic involving sequences of ν digits. This can be decomposed by means of discrete Fourier transforms into a linear form of variates which are asymptotically independent chi-squares, each with one degree of freedom, provided that t, the number of digits in the sample space is prime; but their coefficients are not all ones. However $\nabla\psi_\nu^2=\psi_\nu^2-\psi^2_{\nu-1}$ $(\nu\geqq 1)$ and $\nabla^2\psi_\nu^2=\psi_\nu^2-2\psi^2_{\nu-1}+\psi^2_{\nu-2}$ $(\nu\geqq 2)$ have asymptotic chi-square distributions with $t^{\nu-1}(t-1)$ and $t^{\nu-2}(t-1)^2$ degrees of freedom respectively. *S. W. Nash.*

de Boer, J. Sequential test with three possible decisions for testing an unknown probability. Appl. Sci. Research B. 3, 249–259 (1953).

A three-decision sequential test developed by M. Sobel and A. Wald [Ann. Math. Statistics 20, 502–522 (1949); these Rev. 11, 261] is applied to a binomial problem where the unknown parameter p lies in one of three intervals $(0, a_{12})$, (a_{12}, a_{23}) and $(a_{23}, 1)$. An interesting formula for $E(n|p)$, the expected number of observations required, is developed which makes use of the discrete character of the binomial and the fact that it has been well tabulated. Several numerical comparisons with truncated sequential and with fixed sample size tests are carried out. Not only do the sequential tests show considerable savings but for many problems $\max_p E(n|p)<N$, the number of observations required by the fixed sample size test with the same level of significance. *M. Sobel* (Allentown, Pa.).

Méric, Jean. Test progressif de l'hypothèse que le paramètre d'une loi binomiale est voisin d'une valeur donnée. C. R. Acad. Sci. Paris 237, 1390–1392 (1953).

The problem is the same binomial three-decision problem as in the paper reviewed above but the sequential rule used is different. Again two sequential probability ratio (SPR) tests are carried out but here the rule is to stop only when the sample is simultaneously in the stopping region for both SPR tests. An approximation is given for accepting the "middle" hypothesis; no bound is given for the error involved in this approximation. The O.C. curves for the other two hypotheses are not given. One vague sentence is devoted to $E(n|p)$; no numerical examples or comparisons are given. The approximate formula for h_i in terms of p seems to have the factor $p_i/(1-p_i)$ missing $(i=0, 1)$. *M. Sobel.*

Ramachandran, G., and Ranganathan, J. A non-parametric two sample test. J. Madras Univ. Sect. B. 23, 76–91 (1953).

The sum of squares of the lengths of runs resulting from the arrangement of the observations of two random samples, each of size n, according to size is suggested as a two-sample test criterion. Exact expressions for the first three moments and an asymptotic expression for the fourth moment of this test statistic are computed. On the basis of these moments, a Type VI distribution is fitted for $6\leqq n\leqq 15$. Reviewer's remark: The moments given in the paper imply that the asymptotic distribution of the test statistic is not normal. This is in contradiction with a general theorem by Wolfowitz [Ann. Math. Statistics 13, 247–279 (1942), Th. 4; these Rev. 4, 107]. *G. E. Noether* (Boston, Mass.).

Fraser, D. A. S. Non-parametric theory: Scale and location parameters. Canadian J. Math. 6, 46–68 (1954).

A symmetric function $\phi=\phi(x_1, \cdots, x_n)$ has $E\phi=0$ for every sequence of independent identically distributed variables $x_1, \cdots, x_n$ with an absolutely continuous distribution such that Prob $\{x_i\geqq 0\}=p$ (fixed) if and only if

$$\phi=\sum_i \alpha(x_i)\psi(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),$$

where ψ is an arbitrary bounded symmetric function of $n-1$ variables and $\alpha(x)=p$, $p-1$ for $x\geqq 0$, $x<0$ respectively. This representation is used to obtain a characterization of similar confidence regions for the p-percentile (=location parameter) of a distribution based on a sample of size n, the most powerful similar one-sided confidence region, and the most powerful unbiased confidence region. The one-sided test based on the number of observations exceeding θ is the most powerful test for the hypothesis that the p-percentile is θ against the alternative that it exceeds θ, and the two-sided unbiased test is the most powerful unbiased test against the alternative that the p-percentile is not θ. A scale parameter of a distribution is defined as the difference between two fixed percentiles. No similar confidence region exists for a scale parameter. *D. Blackwell.*

Stuart, Alan. Asymptotic relative efficiencies of distribution-free tests of randomness against normal alternatives. J. Amer. Statist. Assoc. 49, 147–157 (1954).

Under certain conditions Pitman has proposed as a measure of the asymptotic relative efficiency of consistent tests based on statistics t_1 and t_2 the quantity q_1/q_2 where

$$q_i = \frac{\left[\frac{\partial}{\partial\theta}E(t_i)\bigg|_{\theta=\theta_0}\right]^2}{\operatorname{Var}(t_i|\theta=\theta_0)}$$

and θ_0 is the parameter under test. Using q_i as a measure and considering normal regression alternatives $(y_i = \alpha+\beta x_i+\epsilon_i)$, the author calculates the efficiencies of 5 tests of randomness for the hypothesis that the observations are from the same continuous population against the classical test of $\beta=0$. Of the five considered, Kendall's and Spearman's tests rank highest. *H. Teicher* (Lafayette, Ind.).

Weiss, Lionel. A higher order complete class theorem. Ann. Math. Statistics 24, 677–680 (1953).

For decision problems involving a finite number of distributions and a finite number of terminal actions, let $r_{ik}(T)$ denote the expected loss to individual k when the true distribution is F_i, $i=1, \cdots, n$, $k=1, \cdots, s$, and decision procedure T is used. The class of Bayes tests T, i.e. tests which minimize $\sum_{i,k}\xi_{ik}r_{ik}(T)$ for some $\xi=\{\xi_{ik}\}$ with $\xi_{ik}\geqq 0$, $\sum_{i,k}\xi_{ik}=1$, is complete, i.e. if T' is not a Bayes test, there is a Bayes test T with $r_{ik}(T)\leqq r_{ik}(T')$ for all i, k and actual inequality for some i, k. The case $k=1$ is the Wald complete class theorem. An application is that if every decision is rated simply as preferred or not preferred with respect to each alternative and if there is at least one preferred and one not preferred decision for each alternative, then for any test T which is not a Bayes test, there is a Bayes test T' such that for each i the probability of each (not) preferred decision is not smaller (greater) with T' than with T. *D. Blackwell* (Washington, D. C.).

Metakides, Th. A. Calculation and testing of discriminant functions. Trabajos Estadistica 4, 339–368 (1953). (Spanish summary)

This is an expository article on discriminant functions. A computational scheme is also given. *E. Lukacs.*

Hotelling, Harold. New light on the correlation coefficient and its transforms. J. Roy. Statist. Soc. Ser. B. 15, 193–225; discussion, 225–232 (1953).

The density and distribution functions of r, the sample correlation coefficient of correlated normal variates with true correlation ρ, are derived and expressed in terms of rapidly converging series. Convenient asymptotic expansions in powers of n^{-1} are also given, where n is the number of degrees of freedom of r. Recurrence relations and differential equations with respect to n and ρ respectively are derived. Moments of r and $z=\frac{1}{2}\log_e\{(1+r)/(1-r)\}$ are computed. Lastly the author discusses other transformations of r to improve the approximation of the transformed variate to normality or to render its variance more nearly stable and independent of ρ. *S. W. Nash.*

Anderson, R. L. The problem of autocorrelation in regression analysis. J. Amer. Statist. Assoc. 49, 113–129 (1954).

A survey of significance tests for autocorrelation and of various tentative approaches for estimating regression coefficients when the residuals are autocorrelated. *H. Wold.*

Grenander, Ulf, and Rosenblatt, Murray. Comments on statistical spectral analysis. Skand. Aktuarietidskr. 36, 182–202 (1953).

A heuristic presentation of the material of an earlier paper [Ann. Math. Statistics 24, 537–558 (1953); these Rev. 15, 448] together with tables of the relevant limiting distribution and of the function $\pi^{-1}\sin(\pi kn/16)$. The authors would have better achieved their aim of perspicuity had they stated that the test proposed is simply an application of the Kolmogorov goodness-of-fit test to the periodogram. *P. Whittle* (Wellington).

Whittle, P. Estimation and information in stationary time series. Ark. Mat. 2, 423–434 (1953).

The estimating equation for the normalized spectral function and the maximum likelihood estimator of the predictor variance are derived for a stationary Gaussian time series, assuming that both the spectral function and its reciprocal exist on the unit circle. Under mild further restrictions the maximum likelihood estimators of the parameters on which the spectral function depends are shown to be consistent and asymptotically efficient. The covariance matrix of the estimators is expressed in terms of the spectral function, and the estimator of the prediction variance is found to be uncorrelated with the estimators of the other parameters. The last section deals with certain working approximations to the maximum likelihood statistics. *S. W. Nash.*

Middleton, David. Further remarks on the nature of the statistical observer. J. Appl. Phys. 25, 127 (1954).

The author has discussed three classes of optimum statistical procedures for the (incoherent) detection of pulsed carriers in normal random noise [same J. 24, 371–378, 379–391 (1953); these Rev. 14, 1105]. Emphasis is now placed on the operational distinction between observers, and modified comments and conclusions are given.

S. Ikehara (Tokyo).

Middleton, David. Information loss attending the decision operation in detection. J. Appl. Phys. 25, 127–128 (1954).

The detection process in communication theory may be formulated as the problem of testing the statistical hypothesis (H_1) of a signal (in noise) against the (null)-hypothesis (H_0) of noise alone. Application of suitable tests [same J. 24, 371–378, 379–391 (1953); these Rev. 14, 1105] determines the presence or absence of a signal, subject to two classes of error: the Type I error (of probability α), of calling noise a signal when really only noise is present; and the Type II error (probability β) of calling signal and noise when noise alone is present. Since these probabilities are different from zero in practical situations, due to inherent noise and finite observation time, information is necessarily lost when a decision is made. The author proposes a measure of this information loss and discusses its relation to the type of statistical test employed in the detection process.

S. Ikehara (Tokyo).

Wadsworth, G. P., Robinson, E. A., Bryan, J. G., and Hurley, P. M. Detection of reflections on seismic records by linear operators. Geophysics 18, 539–586 (1953).

The authors fit a section of a seismogram with an autocorrelation function, extrapolate it, and judge of the arrival of additional reflected waves by departure of the seismo-

gram from the prediction. They find seismograms to be more extrapolable (except where such new phases appear) than records from various other fields. This effect is due to the fact that the motion on a seismogram actually does consist in large part of trains of well defined waves, instead of stationary noise. The authors' procedure is successful in identifying the arrivals of a number of reflections in the examples which they present, though it is doubtful that the method in its present state can compete with the eye of the skilled seismologist, especially in the accurate estimation of arrival times. *A. Blake* (Catonsville, Md.).

Dugué, Daniel. Statistique et psychologie. Publ. Inst. Statist. Univ. Paris 1, no. 2, 20–40 (1952).

TOPOLOGY

Sonner, H. Die Polarität zwischen topologischen Räumen und Limesräumen. Arch. Math. 4, 461–469 (1953).

The author notes that the topology of a Hausdorff space M may be determined by specifying (1) which sets of M are open, (2) which directed sets of M converge to which points, (3) which filters of M converge to which points. Methods (1) and (2) are related by a polarity described in G. Birkhoff's "Lattice theory" [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, Problem 20, p. 57; these Rev. 10, 673], and studied by B. H. Arnold [Ann. of Math. (2) 54, 319–324 (1951); these Rev. 13, 216]. The author studies a similar polarity between methods (1) and (3), using open sets and filters. On the one hand, the filters converging to p are those containing all open sets containing p.

On the other hand, given a determination having certain properties of what filters converge to what points, one may define a set to be open if it is contained in every filter converging to one of its points. The properties required are then: (i) a filter containing a unit set $\{p\}$ converges to p; (ii) if a filter converges to a point, so does every finer filter; (iii) if a filter does not converge to p, then there is an open set not in the filter, but containing p; and (iv) convergent filters have unique limits. These relations between open sets and filters are expressed by means of a polarity relation. The author also discusses the connection between convergent filters and convergent directed sets; in particular, the residual sets of a directed set converging to p form a basis for a filter converging to p. *O. Frink.*

Gomes, Ruy Luís. Lebesgue space. An example of a regular Riesz space. Gaz. Mat., Lisboa 14, no. 56, 5–6 (1953). (Portuguese)

In the space of bounded real functions on $[a, b]$, take as a basis for open sets all the $\{f;\ u\leqq f\leqq l$ where u is upper- and l is lower-semicontinuous$\}$. The resulting space is presented as an example of a regular Hausdorff space, presumably desirable for some nice distinction of separation axioms. [It is also a zero-dimensional topological group.]
R. Arens (Princeton, N. J.).

Infantozzi, Carlos A. Extensions of a theorem of Riesz to generalized compact sets. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística 2, 121–133 (1953). (Spanish. English summary)

Let x be a space of type $[\mathfrak{V}]$ [cf. Appert and Ky-Fan, Espaces topologiques intermédiaires . . . , Hermann, Paris, 1951; these Rev. 13, 54], and suppose there are two cardinal numbers a and b such that for any $E\subset X$ with $b\leqq \sharp E\leqq a$, E has a complete limit point. It is shown that if $\mathfrak{F}$ is a family of subsets of x, with $\sharp\mathfrak{F}\leqq a$, such that every subset of $\mathfrak{F}$ of power less than b has a common point, then either all $E\in\mathfrak{F}$ have a point in common, or all their derived sets have. This result is related to the well-known equivalence between the definitions of compact Hausdorff space involving complete limit points on the one hand, and filters on the other. The proof is an adaptation of that for the known special case.
R. Arens (Princeton, N. J.).

Gottschalk, W. H. Intersection and closure. Proc. Amer. Math. Soc. 4, 470–473 (1953).

The author considers the question when $\overline{A\cap B}=\bar{A}\cap\bar{B}$ for subsets A,B of a topological space. Let X be a uniform (respectively, uniformizable, i.e., completely regular topological) space, and let $A\subset X$, $B\subset X$. Then A,B are said to be uniformly (continuously) separable provided that $A-U$, $B-U$ are uniformly (continuously) separated whenever U is a neighborhood of $A\cap B$ such that $A\cap B$ and $X-U$ are uniformly (continuously) separated; uniformly (continuously) separated sets are defined in the usual way by means of (uniformly) continuous real-valued functions. It is shown that, in a uniform (uniformizable) space, $\overline{A\cap B}=\bar{A}\cap\bar{B}$ if A,B are uniformly (continuously) separable; if A or B is conditionally compact in X, then $\overline{A\cap B}=\bar{A}\cap\bar{B}$ implies that A,B are uniformly (continuously) separable.
M. Katětov (Prague).

Bing, R. H. A connected countable Hausdorff space. Proc. Amer. Math. Soc. 4, 474 (1953).

A simple example is given of a countable connected Hausdorff space with a countable open base [cf. P. Urysohn, Math. Ann. 94, 262–295 (1925)]. The inductive dimension of the space is 1, the Lebesgue dimension is infinite. It is possible to modify the example to obtain arbitrary (positive) inductive dimension. *M. Katětov* (Prague).

Ellis, Robert. Continuity and homeomorphism groups. Proc. Amer. Math. Soc. 4, 969–973 (1953).

In this paper are given continuity and equicontinuity conditions for abelian groups Φ of homeomorphisms of topological or uniform spaces X. Let Π: $X\times\Phi\to X$ and Π_x: $\Phi\to X$ $(x\in X)$ be the maps such that $(x,\varphi)\Pi=\varphi\Pi_x=x\varphi$, and let $\mathrm{cls}(W)$ denote the closure of W. The main results of the paper may then be stated as follows. Th. 1: Let X be locally compact and let Φ be rigid (i.e., for all $x, y, z\in X$ and $x\neq y$, there exists a neighbourhood U of z such that $\varphi\in\Phi$ and $x\in U\varphi$ imply y non-$\in U\varphi$); then, if Φ is given a first countable topology which makes Π_x continuous for all x, Π is continuous at all (x,φ) such that $\mathrm{cls}(x\Phi)=X$. Th. 3: Let X be a separated uniform space, let each map $\Pi_x^{-1}\Pi_y$: $x\Phi\to y\Phi$ $(x, y\in X)$ be homeomorphic, let Π_x be 1-1 and $\mathrm{cls}(x\Phi)=X$ for all x, and let Φ be "finitely controlled," which means that the topology on Φ of pointwise convergence coincides with the topology of uniform convergence; then Φ is equicontinuous. Th. 4: If X is first-countable compact, the finitely controlled condition may be dropped in Th. 3. Making use of Th. 1, the author also proves that a separable first-countable locally compact T_2-space with an unilaterally continuous abelian group structure is a topological group. *J. L. Tits* (Brussels).

Abdelhay, José. Topological spaces of dimension zero. Revista Cientifica 2, nos. 3–4, 61–71 (1952). (Portuguese)

Let E be a Hausdorff space. E is said to be of dimension zero if for every pair of distinct points $x,y \in E$, there are open sets V_1 and V_2 such that $x \in V_1$, $y \in V_2$, $V_1 \cap V_2 = 0$, and $V_1 \cup V_2 = E$. If E has a Brouwer-Menger-Urysohn or Lebesgue dimension which is equal to zero, then E is of dimension zero; the converse, however, is not true. If E is compact and of dimension zero, then both the Brouwer-Menger-Urysohn and Lebesgue dimension of E are zero.

H. Tong (New York, N. Y.).

Kinoshita, Shin'ichi. A solution of a problem of R. Sikorski. Fund. Math. 40, 39–41 (1953).

In a recent paper [Fund. Math. 37, 213–216 (1950); these Rev. 12, 729] Kuratowski has shown that there exist two 1-dimensional compact subsets of the plane which are not homeomorphic to each other, although each is homeomorphic to a relatively open subset of the other. The present note constructs two 0-dimensional compact sets satisfying the same condition. This answers a question raised by Sikorski [Colloquium Math. 1, 242 (1948); see also Kuratowski, loc. cit.]. *D. W. Hall* (College Park, Md.).

Sorgenfrey, R. H. Dimension lowering mappings of convex sets. Proc. Amer. Math. Soc. 5, 179–181 (1954).

Suppose that $f: K \to F$ is continuous, where K is a compact convex set in Euclidean n-space and F is of dimension m. The results of the paper deal with the size of components of the sets $f^{-1}(y)$. The three main results are as follows. If $n>3$ and $m \leq (n-2)/2$, then some $f^{-1}(y)$ has a component of diameter $\geq w$, the width of K; if $n=3$ and $m=1$, the result also holds. If $n=2$ and $m=1$, then some component of $f^{-1}(y)$ is of diameter $\geq w/3^{\frac{1}{2}}$. If, in addition, K is centrally symmetric, then some $f^{-1}(y)$ has a component of diameter $\geq \frac{1}{2}w3^{\frac{1}{2}}$. Examples are given to show that the latter two results are the best possible. *E. E. Floyd.*

Homma, Tatsuo, and Kinoshita, Shin'ichi. On the regularity of homeomorphisms of E^n. J. Math. Soc. Japan 5, 365–371 (1953).

Let f be a continuous mapping of a metric space X into itself. $\{f^n(A)\}$ is called a bulging sequence if $A \subset X$ and $f^n(A) - \bigcup_0^{n-1} f^i(A) \neq 0$ for all $n>0$. Write $O^+(p) = \bigcup_1^\infty f^n(p)$. Main results: (1) If $A \subset X$ is compact then either $\bigcup_0^\infty f^n(A)$ is compact or $\{f^n(A)\}$ is bulging and $O^+(p) \cap \operatorname{int} A = 0$ for some $p \in A$. (2) If X is locally compact and noncompact then the set of points p such that $(O^+(p))^- \neq X$ is a dense F_σ in X. (3) If X is compact, h is an automorphism of X, and there exist points $a \neq b$ such that $h^n(p) \to a$ for $p \neq b$ and $h^{-n}(p) \to b$ for $p \neq a$, then all integral powers of h are equicontinuous at p, for each $a \neq p \neq b$. In conjunction with a theorem of Kerékjártó [Acta Litt. Sci. Szeged 6, 235–262 (1934)] (3) implies that if h is an automorphism of the plane such that $h^n(p) \to 0$ and $h^{-n}(p) \to \infty$ for all $p \neq 0$, then h is topologically equivalent to the similitude $x' = \frac{1}{2}x$, $y' = \frac{1}{2}y$ or to $x' = \frac{1}{2}x$, $y' = -\frac{1}{2}y$. Result (2) generalizes a theorem of G. D. Birkhoff [Dynamical systems, Amer. Math. Soc. Colloq. Publ., v. 9, New York, 1927, p. 202]; cf. Besicovitch [Proc. Cambridge Philos. Soc. 47, 38–45 (1951); these Rev. 12, 519]. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Oxtoby, John C. Stepanoff flows on the torus. Proc. Amer. Math. Soc. 4, 982–987 (1953).

The author defines a Stepanoff flow as a one-parameter continuous group F of transformations of the torus onto itself such that (1) there exists one and only one stationary point P_0 and (2) the points of each orbit satisfy an equation $y - \alpha x = \text{constant}$, where α is a fixed irrational number. A more restricted class of flows has been considered by V. V. Stepanov [Compositio Math. 3, 239–253 (1936)]. The author gives an example of a flow on the torus which has a unique stationary point P_0 and which is both analytic and area-preserving. This special flow is not a Stepanoff flow but is topologically equivalent to a Stepanoff flow. It is shown that for every flow F on the torus which is topologically equivalent to a Stepanoff flow either (A) the only normalized Borel measure invariant under F is the trivial measure μ_0 for which $\mu_0(P_0) = 1$, or (B) F has one and only one normalized ergodic invariant measure $\mu \neq \mu_0$ and every normalized invariant measure is a linear combination of μ_0 and μ. It is also proved that in case (A) all points of the torus are quasi-regular [N. Kryloff and N. Bogoliouboff, Ann. of Math. (2) 38, 65–113 (1937)] while in case (B) the quasi-regular points form a set of the first category.

Y. N. Dowker (London).

***Reeb, Georges. Sur la nature et la distribution des trajectoires périodiques de certains systèmes dynamiques.** Comptes Rendus du Congrès des Sociétés Savantes de Paris et des Départements tenu à Grenoble en 1952, Section des Sciences, pp. 35–39. Gauthier-Villars, Paris, 1952.

Let a dynamical system be given in a compact n-dimensional manifold V_n. Let there be only a finite number of periodic motions C_i $(i=1, \cdots, N)$; let C_i have characteristic exponents of which r_i have real parts <0 and the remainder have real parts >0; r_i is called the character of C_i. Let every motion approach some C_i as $t \to \pm\infty$ and let the flow near each C_i be topologically equivalent to that determined by the linear theory associated with the characteristic exponents. Let N_r be the number of C_i for which $r_i = r$. It is shown that $N_r + N_{r-1} \geq p_r$, where p_r is the rth Betti number of V_n. It is further stated: if V_n is homologically a sphere and n is odd, then there must be at least one C_i having a given even character $s \leq n-1$; if V_n is the 3-sphere, then $N_2 - N_1 + N_0 = 2$. *W. Kaplan* (Ann Arbor, Mich.).

Klee, V. L., and Utz, W. R. Some remarks on continuous transformations. Proc. Amer. Math. Soc. 5, 182–184 (1954).

"Suppose f is a single-valued transformation of the metric space M onto the metric space fM. Many of the theorems of topology assert that if f is continuous, then it must have various other properties also. We deal here with the problem of determining under what circumstances certain of these properties are actually equivalent to continuity. The properties considered are: (1) fX is compact for every compact $X \subset M$; (2) fY is connected for every connected $Y \subset M$; (3) $f^{-1}q$ is closed for every point $q \in fM$; (4) $f^{-1}\{z: z \in fM$ and $\rho(z,q) = \epsilon\}$ is closed for each $\epsilon > 0$ and $q \in fM$. By an "i-map" on M we will mean a transformation f satisfying the condition (i), where $i = 1, 2, 3$ or 4."

Typical results: "(A) Every 1,3-map on M is continuous." "(B) If M is locally connected at p, then every 1,2-map on M is continuous at p. If M is not locally connected at p, then M admits a real-valued 1,2-map which is discontinuous at p." "(C) For the point $p \in M$, the following assertions are equivalent: (i) M is locally connected at p; (ii) every 2,4-map on M is continuous at p; (iii) every real-valued 2,3-map on M is continuous at p."

Previous work by C. H. Rowe [Bull. Amer. Math. Soc. **32**, 285–287 (1926)] in this connection is referred to in the paper. It was called to the attention of the reviewer by one of the authors that an earlier result related to (C) is due to G. T. Whyburn [ibid. **33**, 185–188 (1927)].

W. H. Gottschalk (Philadelphia, Pa.).

Freudenthal, Hans. Über zwei Probleme von K. A. Sitnikov. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 114–116 (1954).

Sitnikov [Mat. Sbornik N.S. **31**(73), 439–458 (1952); these Rev. **14**, 489] has proved the following theorem. Suppose G is a bounded open set in Euclidean n-space R_n, and suppose that the map $f: G \rightarrow R_n$ has the property that, given $\epsilon > 0$, there exists $\delta > 0$ such that if x is a point of G whose distance from the boundary of G is $< \delta$, then the diameter of $f^{-1}f(x)$ is $< \epsilon$; then fG is open, and its homology groups are isomorphic with those of G. Sitnikov posed the problem of proving that f maps the fundamental group of G isomorphically onto that of fG, and also that G and fG are homeomorphic. The author presents two examples, each of which negates both conjectures. The first of these is elementary, with G the complement of a finite polygon; the second is based upon Antoine's set.

E. E. Floyd.

Gomez Aguilar, Ignacia. Necessary and sufficient condition for the union of two frontier sets to be a frontier set. Revista Mat. Hisp.-Amer. (4) **13**, 328–331 (1953). (Spanish)

The author proves by elementary means: In order that the union of two frontier sets be a frontier set it is necessary and sufficient that one of these two sets have the following property: every point of it which is a limit point of (but not a point of) the other is a limit point of the complement of their union. He makes some simple applications.

F. B. Jones (Chapel Hill, N. C.).

***Ehresmann, Charles. Structures locales.** Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 10, 11 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

This lecture deals with material announced, in part, earlier [C. R. Acad. Sci. Paris **234**, 587–589 (1952); these Rev. **13**, 780]. Axioms are given under which a species of structures (in the sense of Bourbaki) is local. Roughly speaking such structures are determined if one prescribes them locally subject to obvious compatibility conditions (cf. the case of a differentiable structure on a manifold). The local automorphisms of a local structure form a pseudogroup. Conversely a pseudogroup of transformations of a set defines a local structure species: a structure of this species on a set E' is by definition a complete atlas of E onto E' (a maximal collection of 1:1 maps of open sets of E into E', pairwise compatible in an obvious sense, the image sets covering E'). Subpseudogroups lead to subordinated structures. Many examples are given: differentiable, analytic, complex-analytic, algebraic, etc. manifolds; symplectic manifolds; fiber- and leaf-structures. Local isotropy group and local homogeneity are defined, and a brief proof of a theorem of the author [Enseignement Math. **35**, 317–333 (1936), p. 325] is sketched: A compact, simply connected space which is locally homogeneous under a Lie group is a homogeneous space of the Lie group.

H. Samelson (Princeton, N. J.).

Yoneda, Nobuo. On the inverse chain maps. J. Fac. Sci. Univ. Tokyo. Sect. I. **7**, 33–67 (1954).

The desingularization theory of the author [same J. **6**, 393–419 (1953); these Rev. **15**, 147] is applied to maps of manifolds in manifolds to give a rather immediate construction of a chain map inducing the Hopf inverse-homomorphism. Notions generalizing relative manifolds and local coefficients are introduced; a formal homology theory of these objects is studied, and inverse-homomorphisms for certain classes of maps of these objects are constructed.

J. Dugundji (Los Angeles, Calif.).

Samelson, Hans. A connection between the Whitehead and the Pontryagin product. Amer. J. Math. **75**, 744–752 (1953).

The object of this note is to establish a formula which is an important step towards proving the conjectured Jacobi identity for Whitehead products. Let X be an arcwise-connected, simply-connected space, let $[\alpha, \beta] \in \pi_{p+q+1}(X)$ be the Whitehead product of $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$, let Ω be the space of loops in X on the base-point $x_0 \in X$, and let $\tau: \pi_n(X) \rightarrow H_{n-1}(\Omega)$ be the composition of the natural isomorphism $\pi_n(X) \approx \pi_{n-1}(\Omega)$ with the Hurewicz homomorphism $\pi_{n-1}(\Omega) \rightarrow H_{n-1}(\Omega)$.

Now there is a multiplication in Ω, defined by composition of loops. This turns Ω into an H-space and induces a Pontryagin multiplication, $*$, into $H(\Omega) = \sum_n H_n(\Omega)$, compatible with the graduation of $H(\Omega)$. The connection referred to in the title, which holds if $p, q \geqq 1$, is

$$\tau[\alpha, \beta] = (-1)^p(\tau\alpha * \tau\beta - (-1)^{pq}\tau\beta * \tau\alpha). \tag{A}$$

Two proofs are given; the first uses, implicitly, the idea of a universal example for a homotopy construction [see the paper reviewed below], but does not determine the sign $(-1)^p$ in (A); the second proof is elementary, but less obvious.

P. J. Hilton (Cambridge, England).

Blakers, A. L., and Massey, W. S. Products in homotopy theory. Ann. of Math. (2) **58**, 295–324 (1953).

In this paper two generalized Whitehead products are defined and discussed. The former, which was first introduced by W. S. Massey [Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 371–382; these Rev. **13**, 485], associates with elements $\alpha \in \pi_p(A)$, $\beta \in \pi_q(X, A)$ an element $[\alpha, \beta] \in \pi_{p+q-1}(X, A)$. If $(X; A, B)$ is a triad, then the latter product associates with elements $\alpha \in \pi_p(A, A \cap B)$, $\beta \in \pi_q(B, A \cap B)$ an element $[\alpha, \beta] \in \pi_{p+q-1}(X; A, B)$. The properties of these products and their relations to each other, to the original Whitehead product and to a generalized star product (associating, in an obvious way, with $\alpha \in \pi_p(X, A)$, $\beta \in \pi_q(Y, B)$, an element

$$\alpha * \beta \in \pi_{p+q}(X \times Y, A \times Y \cup X \times B))$$

are treated extensively. The proofs of many of these properties are clarified by using the notions of a homotopy construction and of a universal example for homotopy constructions. These ideas are made explicit in the second section of the paper; they are based on Serre's notion of a cohomology construction.

The authors define an n-ary (absolute) homotopy construction for dimensions $p_1, \cdots, p_n$ to q (briefly an n-ary construction) to be a function T which, for any space X and base point $x \in X$, assigns to any elements $\alpha_i \in \pi_{p_i}(X, x)$, $i = 1, \cdots, n$, an element $T(\alpha_1, \cdots, \alpha_n) \in \pi_q(X, x)$; T is required to satisfy the expected naturality condition under

maps $(X, x)\rightarrow(Y, y)$. A universal example for n-ary constructions is a space together with certain elements of its homotopy groups such that any property of such constructions only needs to be verified on the example and such that a construction is determined when its value on the example is known. Thus the universal example for n-ary constructions is the union, K, of n spheres of dimensions $p_1, \cdots, p_n$, with a single common point, together with the elements $\iota_i \in \pi_{p_i}(K)$ represented by the injections $S^{p_i}\rightarrow K$. The set of n-ary constructions is then in (1-1) correspondence with the elements of $\pi_q(K)$. In the same sense, the former generalized Whitehead product is a (relative) homotopy construction for which a universal example consists of the pair $X=E^q\cup S^p$, $A=S^{q-1}\cup S^p$, where E^q is a q-cell attached to S^p at a single point and S^{q-1} is its boundary, together with the elements $\iota \in \pi_p(A)$, $\rho \in \pi_q(X, A)$ represented by the injections of S^p, E^q respectively.

The products are used to prove two theorems. Let $(X; A, B)$ be a triad consisting of a connected CW-complex X and two closed subcomplexes A, B, with $X=A\cup B$. Suppose $A\cap B$ connected and simply-connected, $(A, A\cap B)$ $(m-1)$-connected, $(B, A\cap B)$ $(n-1)$-connected, m, $n>1$. The authors had already proved [Ann. of Math. (2) 55, 192–201 (1952); these Rev. 13, 485] that $(X; A, B)$ is $(m+n-2)$-connected. In this paper they show that $\pi_{m+n-1}(X; A, B)$ contains as a direct summand the isomorphic image, under the generalized Whitehead product, of the tensor product, $\pi_m(A, A\cap B)\otimes\pi_n(B, A\cap B)$. The second theorem discusses the group $\pi_r(S^n; E_+{}^n, E_-{}^n)$ which plays a vital role in studying homotopy groups of spheres. However, the result obtained has been superseded by the work of H. Toda [in the paper reviewed below], who has elucidated $\pi_r(S^n; E_+{}^n, E_-{}^n)$, $r<4n-5$, using the generalized Whitehead products which he introduced independently in an earlier paper [reviewed second below]. *P. Hilton.*

Toda, Hirosi. Topology of standard path spaces and homotopy theory. I. Proc. Japan Acad. 29, 299–304 (1953).

The author uses the following notation. For any topological space X, subspace $A\subset X$, and basepoint $*\in A$, let $\Omega(X, A, *)$ denote the space of all continuous paths in X which start at $*$ and end in A, and $\Omega(X, *)$ the space of closed paths in X at the basepoint $*$. Let $E(X, *)$ denote the suspension of X, i.e., the space obtained from $X\times I$ (where I denotes the closed interval $[0, 1]$) by identifying the subspace $(X\times 0)\cup(X\times 1)\cup(*\times I)$ to a single point which is also denoted by $*$; d: $X\times I\rightarrow E(X, *)$ denotes the identification map. X^n denotes the cartesian product of n copies of X.

For any positive integer n, the author defines a continuous map l_n: $X^n\times A\times I\rightarrow\Omega[E(X), E(A)]$ as follows. Choose a continuous non-negative real-valued function ρ on X such that $\rho(*)=0$ and $\rho(x)\neq 0$ for $x\neq *$. Let $s_0=0$,

$$s_i=s_i(x_1, \cdots, x_n, y)$$
$$=[\rho(x_1)+\cdots+\rho(x_i)]/[\rho(x_1)+\cdots+\rho(x_n)+\rho(y)]$$

for $i=1, \cdots, n$. For $x_1, \cdots, x_n\in X$, $y\in A$, and $t\in I$ let $l_n(x_1, \cdots, x_n; y, t)=f\in\Omega(E(X), E(A))$, where the path f is defined by $f(s)=d[x_i, (s-s_i)/(s_i-s_{i-1})]$ for $i=1, \cdots, n$ and $s_{i-1}\leqq s\leqq s_i$, $f(s)=d[y, t(s-s_n)/(1-s_n)]$ for $s_n\leqq s\leqq 1$. The path f is called a "standard path" and the set of all standard paths is a closed subset of $\Omega(E(X), E(A))$, denoted by $\omega(E(X), E(A))$, called the "standard path space of the pair $(E(X), E(A))$." About this standard path space the author makes the following two assertions. (a) If the pair (X, A) satisfies certain general conditions, the inclusion maps $\omega(E(X))\rightarrow\Omega(E(X))$ and $\omega(E(X), E(A))\rightarrow\Omega(E(X), E(A))$ induce isomorphisms of the homotopy groups in all dimensions. (b) Let X^∞ denote the subspace of the cartesian product of a countable number of copies of X consisting of those points for which all but a finite number of coordinates are the basepoint $*$. Then $\omega(E(X), E(A))$ is homeomorphic to a certain quotient space of X^∞, defined in a rather simple manner. In case X is a cell-complex and A is a subcomplex, the author asserts that $\omega(E(X), E(A))$ can be subdivided into a cell-complex in a rather simple way.

These results are applied to generalize the fundamental theorem of Blakers and Massey on triad homotopy groups [Ann. of Math. (2) 55, 192–201 (1952); these Rev. 13, 485] to a corresponding theorem on the homotopy groups of an n-ad. The author also studies the homotopy groups of spheres, $\pi_{n+k}(S^n)$, for $k\leqq 8$. The results obtained are contained in those announced by J. P. Serre [C. R. Acad. Sci. Paris 236, 2475–2477 (1953); these Rev. 14, 1110].

In this note, the author does not even give a hint of the method of proof of his fundamental theorems. Moreover, there are so many misprints and errors, and the exposition is so condensed, that it is often difficult to know what the author had in mind. *W. S. Massey* (Providence, R. I.).

Toda, Hirosi. Generalized Whitehead products and homotopy groups of spheres. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 3, 43–82 (1952).

L'un des buts de ce travail est la détermination de certains groupes d'homotopie de sphères $\pi_{n+i}(S^n)$. Des résultats semblables ont été obtenus indépendamment par P. Hilton [Proc. London Math. Soc. (3) 1, 462–493 (1951); Proc. Cambridge Philos. Soc. 48, 547–554 (1952); ces Rev. 13, 674; 14, 306] et J. P. Serre [C. R. Acad. Sci. Paris 234, 1340–1342 (1952); ces Rev. 13, 675]. Le calcul de chaque groupe $\pi_{n+i}(S^n)$ est accompagné d'une construction géométrique explicite de ses générateurs, au moyen de constructions de Hopf, suspensions et compositions. Les groupes calculés sont les suivants: $\pi_{n+3}(S^n)$ pour tout n; les groupes stables $\pi_{n+i}(S^n)$ pour $4\leqq i\leqq 8$. Pour $i=7, 8$ la détermination est incomplète, mais ces groupes sont maintenant connus [Serre, ibid. 236, 2475–2477 (1953); ces Rev. 14, 1110; Toda, note analysée ci-dessus]. Dans un appendice (App. 2) rajouté après la publication de la première note de Serre citée, l'auteur calcule $\pi_{n+4}(S^n)$ et $\pi_{n+5}(S^n)$ pour tout n, et donne une détermination partielle de $\pi_{n+6}(S^n)$ pour $n\geqq 5$.

Le principe de la méthode est le suivant: l'auteur cherche à construire explicitement un CW-complexe K_n ayant un nombre fini de cellules en toute dimension, tel que le squelette $K_n{}^{n+1}$ soit une sphère S^n et que $\pi_i(K_n{}^{n+k})=0$ pour $n<i<n+k$. La construction de K_n se fait par l'adjonction successive de cellules de dimension $n+k$, en nombre minimum; chaque adjonction suppose qu'on a auparavant déterminé certains groupes d'homotopie. Toda utilise la détermination effective des groupes d'Eilenberg-MacLane $H_{n+k}(Z, n)$ faite par ces auteurs pour les petites valeurs de k, et le fait que le conoyau de

$$\pi_{n+k}(K_n{}^{n+k-1}, K_n{}^{n+k-2})\rightarrow\pi_{n+k-1}(K_n{}^{n+k-2})$$

est isomorphe à $H_{n+k}(Z, n)$. L'auteur n'utilise pas la suite spectrale, mais fait des constructions géométriques; dans ce but, il consacre d'abord 5 chapitres à une exposition extrêmement concise mais très complète des opérations et constructions plus ou moins classiques en théorie de l'homotopie, avec des compléments et développements nouveaux que nous allons brièvement signaler.

Chap. 1: extension aux petites dimensions du "théorème d'excision" de Blakers-Massey relatif aux triades (th. 1.23 et 1.26). Chap. 2: excellente exposition des produits de Whitehead, y compris leur généralisation aux triades [trouvée indépendamment par Blakers et Massey, Bull. Amer. Math. Soc., **57**, 142 (1951), abstract 165t]; le produit de Whitehead des triades et le produit de Whitehead "relatif" sont en connexion étroite avec la "construction de Hopf" qui, à $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$ tels que $[\alpha, \beta]=0$, associe un élément de $\pi_{p+q+1}(E(X))$, $E(X)$ désignant l'espace déduit de X par "suspension." Au chap. 3, l'auteur définit un "homomorphisme de Hopf" $H: \pi_p(S^r; E_+^r, E_-^r)\rightarrow\pi_{p+1}(S^{2r})$ ($p\geqq 3$); d'où, par composition avec $\pi_p(S^r)\rightarrow\pi_p(S^r; E_+^r, E_-^r)$, l'homomorphisme $H_0: \pi_p(S^r)\rightarrow\pi_{p+1}(S^{2r})$ de Hilton (1er article cité, p. 473), qui, pour $p\leqq 4r-4$, satisfait à $H_0=E\circ H'$, où $H': \pi_p(S^r)\rightarrow\pi_p(S^{2r-1})$ désigne l'homomorphisme de G. W. Whitehead-Hilton, et $E: \pi_p(S^{2r-1})\rightarrow\pi_{p+1}(S^{2r})$ désigne la suspension. Les propriétés de H et H_0 sont étudiées. La définition de H est ensuite généralisée au cas suivant: soit un espace X^*, un sous-espace fermé X tel que X^*-X soit réunion de cellules ouvertes de dimension r, disjointes, et soit $\mathcal{E}^r$ la réunion des adhérences de ces cellules, $\dot{\mathcal{E}}^r$ la frontière de $\mathcal{E}^r$; l'auteur définit un homomorphisme H: $\pi_p(X^*; \mathcal{E}^r, X)\rightarrow\pi_{p+1}(E^r(X))$, où E^r désigne la suspension itérée; il en déduit (th. 3.13) que si $(X, \dot{\mathcal{E}}^r)$ est "smooth" et m-connexe, $\pi_p(X^*; \mathcal{E}^r, X)$ admet un facteur direct isomorphe à $\pi_{p-r+1}(X, \dot{\mathcal{E}}^r)\otimes\pi_r(\mathcal{E}^r, \dot{\mathcal{E}}^r)$ lorsque

$$p\leqq r+m+\inf(m, r)-1.$$

Au chap. 4, l'auteur détermine certains éléments $\neq 0$ des $\pi_n(S^r)$. Notations: ι_n générateur canonique de $\pi_n(S^n)$; η_n générateur de $\pi_{n+1}(S^n)$ ($n\geqq 2$); pour $n\geqq 3$, $\alpha_n \in \pi_{n+3}(S^n)$ est le suspendu de $\alpha_3 \in \pi_6(S^3)$, obtenu par la construction de Hopf appliquée à η_2 et ι_2; pour $n\geqq 4$, $\nu_n \in \pi_{n+3}(S^n)$ est le suspendu de $\nu_4 \in \pi_7(S^4)$, obtenu par la construction de Hopf appliquée à ι_3 et ι_3, ou encore par la fibration de Hopf $S^7\rightarrow S^4$. On a $2\nu_4-\alpha_4=[\iota_4, \iota_4]$, $2\nu_n-\alpha_n=0$ ($n\geqq 5$); $\eta_n\circ\nu_{n+1}=\alpha_n\circ\eta_{n+3}$ ($\neq 0$ pour $n=3, 4$; nul pour $n\geqq 5$); $\nu_n\circ\eta_{n+3}\neq 0$ pour $n=4, 5$, nul pour $n\geqq 6$; $2\nu_4\circ\nu_7=[\iota_4, \iota_4]$, et, pour $n\geqq 5$, $2\nu_n\circ\nu_{n+3}=0$, $\nu_n\circ\nu_{n+3}\neq 0$.

Au chap. 5, dans le but d'obtenir un élément $\zeta \in \pi_{n+3}(S^n)$ d'ordre 4 pour $n\geqq 3$ [cf. Barratt et Paechter, Proc. Nat. Acad. Sci. U. S. A. **38**, 119–121 (1952); ces Rev. **13**, 674] tel que $2\zeta=\eta_n\circ\eta_{n+1}\circ\eta_{n+2}$, l'auteur définit deux constructions: l'une donne un homomorphisme T:

$$_2(\pi_n(X))\rightarrow\pi_{n+1}(X)\otimes Z_2$$

et on montre que $T(\alpha)$ est la classe de $\alpha\circ\eta_n$; l'autre associe à un triple $\alpha \in \pi_r(S^s)$, $\beta \in \pi_m(S^r)$, $\gamma \in \pi_n(S^m)$ tel que $\alpha\circ\beta=0$ et $\beta\circ\gamma=0$, un élement $\{\alpha, \beta, \gamma\}$ du quotient de $\pi_{n+1}(S^s)$ par le sous-groupe $\alpha\circ\pi_{n+1}(S^r)+\pi_{m+1}(S^s)\circ E(\gamma)$. Aux chap. 7 et 8, et dans l'App. 2, les générateurs suivants sont explicités:

α_3 pour $\pi_6(S^3)\approx Z_{12}$
ν_4 et α_4 pour $\pi_7(S^4)\approx Z+Z_{12}$
ν_n pour $\pi_{n+3}(S^n)\approx Z_{24}$ ($n\geqq 5$)
$\eta_3\circ\nu_4$ pour $\pi_7(S^3)\approx Z_2$
$\eta_4\circ\nu_5$ et $\nu_4\circ\eta_7$ pour $\pi_8(S^4)\approx Z_2+Z_2$
$\nu_5\circ\eta_8$ pour $\pi_9(S^5)\approx Z_2$
$\eta_3\circ\nu_4\circ\eta_7$ pour $\pi_8(S^3)\approx Z_2$
$\eta_4\circ\nu_5\circ\eta_8$ et $\nu_4\circ\eta_7\circ\eta_8$ pour $\pi_9(S^4)\approx Z_2+Z_2$
$[\iota_6, \iota_6]$ pour $\pi_{11}(S^6)\approx Z_2$
$\nu_n\circ\nu_{n+3}$ pour $\pi_{n+6}(S^n)\approx Z_2$ ($n\geqq 5$).

Signalons l'appendice 1, consacré au calcul des premiers groupes d'homotopie d'un espace obtenu en attachant une $(n+1)$-cellule à une n-sphère par une application de degré 2 de son bord. Dans l'app. 2, utilisant un résultat de Serre, l'auteur donne une suite exacte qui n'est autre que celle établie indépendamment par G. W. Whitehead [Ann. of Math. (2) **57**, 209–228 (1953); ces Rev. **14**, 1110].

H. Cartan (Paris).

Nakaoka, Minoru. Classification of mappings of a complex into a special kind of complex. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **3**, 101–143 (1952).

Soit Y un espace tel que $\pi_i(Y)=0$ pour $0\leqq i<n$ et $n<i<n+k$. Soient K un complexe, L un sous-complexe, f et g deux applications continues $K^n\cup L\rightarrow Y$ (K^n: squelette de dimension n), qui coïncident sur L et se prolongent en $K^{n+1}\cup L\rightarrow Y$. La différence des "secondes obstructions" de f et g est un élément $\mu^{n+k+1} \in H^{n+k+1}(K, L; \pi_{n+k}(Y))$, qu'il s'agit de calculer en fonction de la première obstruction $\lambda^n \in H^n(K, L; \pi_n(Y))$. La solution de ce problème est classique pour $k=1$ (Steenrod, Postnikov, J. H. C. Whitehead); pour k quelconque, elle se ramène au calcul de l'invariant $\mathbf{k} \in H^{n+k+1}(\pi_n(Y), n; \pi_{n+k}(Y))$ d'Eilenberg-MacLane à l'aide de la classe fondamentale $\mathbf{i} \in H^n(\pi_n(Y), n; \pi_n(Y))$, tout au moins lorsque $n\geqq k+2$ [cf. Eilenberg et MacLane, Proc. Nat. Acad. Sci. U. S. A. **38**, 325–329 (1952); ces Rev. **13**, 966].

Le cas $k=2$, $n\geqq 3$ a été étudié par Shimada et Uehara [Nagoya Math. J. **4**, 43–50 (1952); ces Rev. **13**, 859]. L'auteur donne ici une solution complète pour $k=2$ et $k=3$ ($n\geqq 4$), en supposant que Y soit un complexe fini (restriction probablement inessentielle). L'expression explicite de μ^{n+k+1} en fonction de λ^n fait intervenir, dans le "cas stable" ($n\geqq k+2$), des puissances de Steenrod itérées (relatives aux entiers premiers $p=2, 3$) et des homomorphismes naturels $\pi_n(Y)\otimes Z_p\rightarrow\pi_{n+k}(Y)$, resp. $_p(\pi_n(Y))\rightarrow\pi_{n+k}(Y)$ qui sont explicitement définis (on note $_pG$ le sous-groupe des éléments d'ordre p d'un groupe abélien G). Dans le cas non stable $k=3$, $n=4$, il intervient aussi un cup-carré et un carré de Pontrjagin. Les mêmes formules donnent aussi la solution du problème de classification homotopique des applications "normales" $K^{n+1}\cup L\rightarrow Y$; cependant, dans le cas non stable ($k=3$, $n=4$), le cup-carré et le carré de Pontrjagin sont remplacés par un "carré de Postnikov."

Pour démontrer ses résultats, l'auteur étudie, comme Toda [travail analysé ci-dessus], des CW-complexes $R_n^k(h)$ ($h=0$ ou une puissance d'un entier premier), de dimension $n+k$, tels que $\pi_i(R_n^k(h))=0$ pour $0\leqq i<n$ et $n<i<n+k$, $\pi_n(R_n^k(h))=Z_h$ (groupe cyclique d'ordre h). Il détermine explicitement les puissances de Steenrod et les carrés de Pontrjagin et Postnikov de ces espaces, ainsi que $\pi_q(R_n^k(h))$ pour $q\leqq n+3$ ($k=1, 2, 3$; $h=0, 2^f$ ou 3^f). *H. Cartan.*

Nakaoka, Minoru. On homotopy classification and extension. Proc. Japan Acad. **29**, 6–9 (1953).

Nous conservons les notations du compte rendu précédent. L'auteur donne, sans démonstration, l'expression complète de μ^{n+k+1} en fonction de λ^n dans le cas stable ($n\geqq k+2$), pour $k\leqq 6$ (les cas $k=4, 5, 6$ sont donc nouveaux). Cela fait encore intervenir des puissances de Steenrod itérées et des homomorphismes (1) $\pi_n(Y)\otimes Z_p\rightarrow\pi_{n+k}(Y)$, resp. (2) $_p(\pi_n(Y))\rightarrow\pi_{n+k}(Y)$, qui sont explicitement décrits à l'aide de certains générateurs des groupes d'homotopie $\pi_{n+k}(R_n^k(0))$, resp. $\pi_{n+k}(R_n^k(p))$.

Note du rapporteur: il existe visiblement une formule générale pour tout k; les homomorphismes des types (1) et (2) peuvent être définis en utilisant l'homomorphisme connu

$$H_{n+k+1}(\pi_n(Y), n)\rightarrow\pi_{n+k}(Y)$$

défini par l'invariant d'Eilenberg-MacLane [voir aussi Cartan et Serre, C. R. Acad. Sci. Paris **234**, 393–395 (1952), prop. 2; ces Rev. **13**, 675], et se servant de la détermination explicite des groupes stables d'Eilenberg-MacLane $H_{n+k+1}(\pi, n)$, $n \geqq k+2$. *H. Cartan* (Paris).

Whitehead, J. H. C. On the $(n+2)$-type of an $(n-1)$-connected complex $(n \geqslant 4)$. Proc. London Math. Soc. (3) **4**, 1–23 (1954).

The $(n+2)$-type of an $(n-1)$-connected complex $(n \geqq 4)$ is given by the triple (π_n, π_{n+1}, η) where $\eta: \pi_n \to \pi_{n+1}$ is the homomorphism induced by an essential map $S^{n+1} \to S^n$; in case $\pi_i(K)=0$ for all $i \neq n$, $n+1$, the $(n+2)$-type characterizes its homotopy type. Let K be an $(n-1)$-connected CW complex $(n \geqq 4)$. Denote by $H_p(2)$ its mod 2 homology group and define the homomorphism $\mathfrak{h}: H_{n+1}(2) \to H_n/\eta(2\pi_{n+1})$ by first sending each mod 2 cycle z into the cycle $\frac{1}{2}\partial z$ and then projecting onto the factor group. The author shows that $\mathfrak{h}$ can also be defined as follows: For any additive abelian group G with $2G=0$, $H^p(K, G) \approx \mathrm{Hom}\,(H_p(2), G)$, so that $u \in \mathrm{Hom}\,(H_p(2), G)$ can be regarded as an element of $H^p(K, G)$ and $\mathrm{Sq}_{p-i}u$ as an element of $\mathrm{Hom}\,(H_{p+i}(2), G)$. The author constructs a homomorphism u:

$$H_n(2) \to H_n/\eta^{-1}(2\pi_{n+1}) = G$$

with $\mathfrak{h} = \mathrm{Sq}_{n-1}u$.

If G, Q are any two additive abelian groups with $2G=2Q=0$, the author constructs an isomorphism Sq of Hom (Q, G) onto the group of abelian group extensions of G by Q; using this with $Q=H_{n+1}(2)$, $G=H_n/\eta^{-1}(2\pi_{n+1})$, he shows that the image of $\pi_{n+2}(K^{n+1})$ under injection into $\pi_{n+2}(K^{n+2})$ belongs to Sq $\mathfrak{h}$. From the theory given it then follows that if $\pi_i(K)=0$ for $i \neq n$, $n+1$ and $n \geqq 4$, then $H_{n+1}=\pi_{n+1}/\eta\pi_n$, $H_{n+2}=\eta^{-1}(0)/2\pi_n$ and $H_{n+2} \in \mathrm{Sq}\,\mathfrak{h}$. When Sq $\mathfrak{h}$ is known, a group in the class Sq $\mathfrak{h}$ can be exhibited, and the structure of the homology groups calculated in a finite number of steps whenever π_n, π_{n+1} are finitely generated and η is given. *J. Dugundji* (Los Angeles, Calif.).

Whitehead, J. H. C. The G-dual of a semi-exact couple. Proc. London Math. Soc. (3) **3**, 385–416 (1953).

A "semi-exact couple" is defined to be a pair of abelian groups, A and C together with homomorphisms $d: C \to A$ and $j: A \to C$ which satisfy the condition $d \circ j = 0$. Given such an exact couple, one can define an endormorphism ∂: $C \to C$ by $\partial = j \circ d$. Then $\partial^2=0$, and we may form the derived group $H=H(C)$. Let Γ denote the kernel of j, and Π the cokernel of d (i.e., $\Pi = A/d(C)$). The homomorphisms d and j induce homomorphisms d_*: $H \to \Gamma$ and j_*: $\Pi \to H$; also, the identity map $A \to A$ induces a homomorphism i_*: $\Gamma \to \Pi$. These three groups and homomorphisms fit together to form a triangular diagram, $H \to \Gamma \to \Pi \to H$, which is exact at two of the three groups. However, it is not generally true that kernel $d_* = \mathrm{image}\ j_*$; only the weaker condition $d_* \circ j_* = 0$ holds.

This algebraic scheme is a generalization of a scheme previously introduced by the same author [Ann. of Math. (2) **52**, 51–110 (1950); these Rev. **12**, 43]. The only change is the replacement of the condition "kernel $d = \mathrm{image}\ j$" by the weaker condition "$d \circ j = 0$." As a consequence, the triangular diagram $H \to \Gamma \to \Pi \to H$ is no longer exact at H.

Let G be any abelian group. The "G-dual" of A, denoted by $A^\#(G)$, is defined to be the group of all homomorphisms $A \to G$; similarly, the G-duals of the homomorphisms d: $C \to A$ and j: $A \to C$ are the naturally induced homomorphisms $d^\#: A^\#(G) \to C^\#(G)$ and $j^\#: C^\#(G) \to A^\#(G)$. Using these definitions, it is readily seen that the G-dual of a semi-exact couple is again a semi-exact couple. Note, however, that even if the condition "kernel $d = \mathrm{image}\ j$" holds, we cannot conclude that the condition "kernel $j^\# = \mathrm{image}\ d^\#$" holds in the G-dual. Naturally all these definitions still hold with only slight modifications in case A, C, and G are modules over a given group or ring of operators, and we require that all homomorphisms involved commute with the operators.

The author applies this algebraic scheme to the following topological situation. Let K be a connected cell complex, and let $A = \sum A_n$, $C = \sum C_n$ be graded groups defined by $A_n = \pi_n(K^n)$, $C_n = \pi_n(K^n, K^{n-1})$ for $n > 2$; for $n \leqq 2$, these definitions have to be appropriately modified (here K^n denotes the n-skeleton of K). Let d and j be the homomorphisms induced by the homotopy boundary operator and the inclusion maps $K^n \to (K^n K^{n-1})$, $n = 1, 2, 3, \cdots$, respectively; with these definitions, the pair of groups (A, C) form a semi-exact couple. It then turns out, as is well known, that the graded groups H and Π obtained by the above scheme are the integral homology group of the universal covering space of K, and the homotopy group of K respectively. Let G be an arbitrary coefficient group, and form the G-dual of this semi-exact couple. If we apply the above described algebraic process to the semi-exact couple thus obtained, there results a triangular diagram $H^*(G) \to \Gamma^*(G) \to \Pi^*(G) \to H^*(G)$ which is exact at two of the three places. It turns out that $H^*(G)$ is the cohomology group of K with coefficients in the group G, and $\Gamma^*(G)$ is a graded group whose q-dimensional component is the group of all homomorphisms $\pi_q(K) \to G$. The author makes various topological applications of the groups and homomorphisms in this triangular diagram, most of which cannot be stated without going into long details. In the case where one takes account of the operations of the fundamental group of K, and assumes that the coefficient group also admits the fundamental group as a group of operators, connections are made with the work of Eilenberg and MacLane on spaces with operators [cf., e.g., Trans. Amer. Math. Soc. **65**, 49–99 (1949); these Rev. **11**, 379]. *W. S. Massey* (Providence, R. I.).

Hilton, P. J. A certain triple Whitehead product. Proc. Cambridge Philos. Soc. **50**, 189–197 (1954).

If X is a topological space, one may make the direct sum of the homotopy groups of X, $\sum \pi_n(X)$ $(n \geqq 2)$, into a graded, non-associative ring by use of the product of J. H. C. Whitehead. Not much is known about the rings thus obtained. In this paper, the author studies one of the simplest and most important non-trivial cases, viz., the case where $X = S^n$, the n-sphere. The results obtained are as follows. Let ι_n denote a generator of the group $\pi_n(S^n)$. If n is odd, or if $n=2$, it is readily shown that the triple product $[[\iota_n, \iota_n], \iota_n]=0$. The author proves that if n is an even integer >2, then there exists an element $\alpha \in \pi_{3n-3}(S^{n-1})$ such that $[[\iota_n, \iota_n], \iota_n]$ is the suspension of α. More precise results are obtained in case $n=4$. In this case, it is shown that $[[\iota_4, \iota_4], \iota_4] \neq 0$, and all products are determined in S^4 up to the dimension 13; in particular, it is shown that all products are zero in $\pi_q(S^4)$ for $q = 11$, 12, or 13. *W. S. Massey* (Providence, R. I.).

Hilton, P. J. On the Hopf invariant of a composition element. J. London Math. Soc. **29**, 165–171 (1954).

The author had previously defined a homomorphism H^+: $\pi_r(S^n) \to \pi_{r+1}(S^{2n})$ for all r, n [Proc. London Math. Soc. (3) **1**, 462–493 (1951); these Rev. **13**, 674]. Let $\beta \in \pi_p(S^n)$

$\gamma \in \pi_r(S^p)$ and let "$\circ$" be the composition operation (so that $\beta \circ \gamma \in \pi_r(S^n)$); it is shown that

$$H^+(\beta \circ \gamma) = H^+(\beta) \circ E\gamma + E^n\beta \circ E^p\beta \circ H^+(\gamma),$$

where E^r denotes r-fold suspension. This formula can be regarded as a generalization to all values of r, n of results previously proved by G. W. Whitehead and the author for very restricted ranges of r, p, n. As a corollary, it follows that when $n=2, 4, 8$ and $H_1: \pi_r(S^n) \to \pi_r(S^{2n-1})$ is the projection, then $2(H^+(\alpha) - EH_1(\alpha)) = 0$ for every $\alpha \in \pi_r(S^n)$, and that $H^+(\alpha) - EH_1(\alpha) = E^n\beta \circ E^{2n-1}\beta \circ H^+H_1(\alpha)$, where $\beta \in \pi_{2n-1}(S^n)$ is the Hopf class. *J. Dugundji.*

Massey, W. S. Products in exact couples. Ann. of Math. (2) 59, 558–569 (1954).

In this paper, multiplication is introduced into exact couples, thus enhancing their value in algebraic topology, e.g., when applied to the singular cohomology of a fibre-space.

Let $\langle A, C; f, g, h\rangle$ be an exact couple in which C is a ring, let $d=gh$, and let α be an involution of C anti-commuting with d. Let μ_n, $n \geqq 0$, be the condition that if $h(x)=f^n(a)$, $h(y)=f^n(b)$, $x, y \in C$, $a, b \in A$, then there exists $c \in A$ such that $h(xy)=f^n(c)$, and $g(c)=g(a)\cdot y+(\alpha x)\cdot gb$. Then μ_0 is just the condition that (C, d, α) be a differential ring. Let $\langle A', C'; f', g', h'\rangle$ be the derived couple, and let $d'=g'h'$. Then C' is a ring, and α induces an involution α' of C' anti-commuting with d'. It turns out that μ_n holds in $\langle A, C\rangle$, $n \geqq 1$, if and only if μ_{n-1} holds in $\langle A', C'\rangle$, granted that μ_0 holds in $\langle A, C\rangle$. Thus if μ (μ_n for all $n \geqq 0$) holds in $\langle A, C\rangle$ it holds in all the derived couples.

If A is a ring, if g is a multiplicative homomorphism, and if $f(xy)=xf(y)=(fx)y$ (f is a transducer, according to the author), then a new multiplication may be introduced into A' whereby g' is multiplicative and f' is a transducer. Finally, a condition ν on the pair A, C is discussed which is transferred to all the derived couples in the presence of condition μ. It is shown that all the conditions are verified if $\langle A, C\rangle$ is the exact couple associated with a filtered differential ring.

The last section gives conditions for exact couples to give rise to convergent spectral sequences and is not explicitly concerned with products.

An unusually large number of proofs are left to the reader, but this seems justified. On p. 565, $\sum_{q,p} C_{p,q}$ should read $\sum_{p,q} C_{p,q}$. *P. J. Hilton* (Cambridge, England).

GEOMETRY

Lorent, H. Sur un ensemble de triangles rectangles du plan ou de l'espace. Anais Fac. Ci. Porto 37, 5–36 (1953).

Thébault, Victor. Pascal hexagons associated with a triangle. Amer. Math. Monthly 61, 328–330 (1954).

Klepper, W. Über die natürliche Gleichung

$$R(s) = \mu a\left(\lambda + \cos\frac{s}{a}\right).$$

Elemente der Math. 9, 56–63 (1954).

Riabokin, Toma. On real unicursal curves. Department of Aeronautical Engineering, University of Minnesota, Minneapolis, Minn., Research Rep. No. 86, vii+124 pp. (1952).

The author gives in this paper an elaborate method of constructing unicursal curves, based on considerations of analytical geometry, and seems to ignore that the same results, together with the equation of the curve, can easily be obtained by topological "small variations," starting from a curve decomposed into elementary curves [see, e.g., Hilton, Plane algebraic curves, Oxford, 1920]. With this method it is easy to construct algebraic curves with nodes or cusps and other singularities. *M. Piazzolla Beloch.*

Petronijević, Branislav. Application des fonctions hyperboliques à la réduction des formules trigonométriques du triangle rectangle dans le plan de Lobatschewsky. Srpska Akad. Nauka. Zbornik Radova 35. Mat. Inst. 3, 289–299 (1953). (Serbo-Croatian. French summary)

A slight simplification of the Liebmann-Sommerville procedure for deriving hyperbolic trigonometry.
H. Busemann (Copenhagen).

Permutti, Rodolfo. Spazi affini generalizzati e relative proprietà reticolari. Ricerche Mat. 2 (1953), 192–203 (1954).

A generalized affine space S^* is obtained from a projective space S of finite dimension n in the same way as an ordinary affine space except that instead of excluding a hyperplane "at infinity," one excludes the set union of a set of subspaces S_i whose dimension is not restricted except that the remaining set S^* must have n independent points. Many properties of ordinary affine spaces are found to be still true in S^*. The subspaces of S^* necessarily form an upper semi-modular relatively complemented (i.e., matroid) lattice whose length equals the dimension n of S^*. For $n=3$ [points are assigned dimension 1], these conditions are also sufficient for a lattice to be the lattice of subspaces of a generalized affine plane, which can essentially be constructed from the points of the lattice. The ordinary affine plane is characterized by adjoining the parallel postulate. *P. M. Whitman.*

Sasaki, Usa, and Fujiwara, Shigeru. The characterization of partition lattices. J. Sci. Hiroshima Univ. Ser. A. 15, 189–201 (1952).

O. Ore has characterized the lattice L of all partitions of a set in geometric terms [Duke Math. J. 9, 573–627 (1942); these Rev. 4, 128]. In this paper the authors give first a somewhat different geometric characterization of L as the lattice of all subgeometries of a geometry with the properties: 1) a line contains at most three points; 2) two distinct points determine a line; 3) a plane contains either three, four, or six points; 4) if a line has one parallel, it has exactly two parallels; 5) a line with only two points on it has always at least one parallel line. Here a plane is determined by three non-collinear points, and parallel lines as usual are coplanar lines without common point.

The authors next define a general partition lattice abstractly as a planar matroid lattice satisfying three conditions L_1, L_2, and L_3 which are related to the above geometrical properties. By a matroid lattice is meant, as usual, a lattice which is relatively atomic, upper continuous, and semi-modular. A planar lattice is one such that if p and q are atoms, and $a \neq 0$ and $q \leqq p \cup a$, then there exist atoms r and s such that $r \cup s \leqq a$ and $q \leqq p \cup r \cup s$. They then prove that every general partition lattice thus defined is isomorphic to the direct union of lattices of partitions of sets,

and that a general partition lattice is isomorphic to the lattice of all partitions of some set if and only if it is irreducible (not a direct union). *O. Frink.*

Sasaki, Usa. Lattice theoretic characterization of geometries satisfying "Axiome der Verknüpfung." J. Sci. Hiroshima Univ. Ser. A. 16, 417–423 (1953).

The author has previously characterized [see preceding review] the lattice of all subspaces of an affine space, that is, of a space satisfying Hilbert's axioms of incidence and the parallel postulate, but without restrictions of dimensionality. In the present paper he characterizes the lattice of subspaces of a geometry which satisfies merely the incidence axioms, without any assumption about parallels. These incidence axioms are, in brief: 1) two distinct points determine a line; 2) three non-collinear points determine a plane; 3) if two planes in the same 3-space have a common point, they have a common line. Let us call such a space an incidence geometry.

The author calls a lattice L strongly plane if for any atoms p, q, and r of L, and any element a of L such that $p\leqq q\cup a$ and $r\leqq a$, there exists an atom $s\leqq a$ of L such that $p\leqq q\cup r\cup s$. He then proves that a lattice is isomorphic with the lattice of all subspaces of an incidence geometry, if and only if it is relatively atomic, upper continuous, semimodular, and strongly plane. He also gives a slightly different characterization of the lattice of subspaces of an affine space, using a condition of Wilcox related to semimodularity. *O. Frink* (State College, Pa.).

Maeda, Fumitomo. Lattice theoretic characterization of abstract geometries. J. Sci. Hiroshima Univ. Ser. A. 15, 87–96 (1951).

After defining a very general type of abstract geometry, the author characterizes in lattice-theoretical terms the lattice of all subgeometries of such a system. This provides a generalization of results of Prenowitz [Ann. of Math. (2) 49, 659–688 (1948); these Rev. 10, 57], G. Birkhoff, and the reviewer [Trans. Amer. Math. Soc. 64, 299–316 (1948); these Rev. 10, 279], characterizing the lattices of linear subspaces of a projective geometry and of convex sets of a descriptive geometry. The author defines an abstract geometry with finitary operations to be a system of elements called points, together with an operation of addition of any finite number of points to form a point set, the join of the finite set. This operation is assumed to obey a rather weak set of postulates.

The principal theorem is that a lattice is isomorphic to the lattice of all subgeometries of a suitable abstract geometry with finitary operations if and only if it is relatively atomic and upper continuous. A lattice is relatively atomic if every element is a join of atoms, and upper continuous if it is complete and if $x_\alpha\uparrow x$ implies $x_\alpha\cap y\uparrow x\cap y$. It is shown that in a complete relatively atomic lattice, upper continuity is equivalent to the condition that an atom is contained in the join of a set of atoms only if it is contained in the join of a finite subset.

The author also generalizes the notions of perspectivity and projectivity of atoms, previously studied in modular lattices, to more general lattices. He defines p perspective to q to mean there is an x such that $q\leqq p\cup x$, and $q\cap x=0$. He then shows that a relatively atomic, upper continuous lattice may be decomposed into a direct sum of sublattices so that two atoms are in the same sublattice if and only if they are projective. *O. Frink* (State College, Pa.).

Nordhaus, E. A., and Lapidus, Leo. Brouwerian geometry. Canadian J. Math. 6, 217–229 (1954).

Definitions and notations. L denotes a lattice with a least element O. For any three elements a, b, c of L, b is lattice between a and c if $ab+bc=b=(a+c)(b+c)$. An L-metrized space is an abstract set S of elements such that with each pair x, y of elements of S there is associated an element $d(x, y)$ of L, the L-distance, satisfying the usual distance axioms in their lattice interpretation. For any three elements x, y, z of S, y is metrically between x and z if $d(x, y)+d(y, z)=d(x, z)$. The space S is ptolemaic if the three products of the pairs of opposite L-distances for each four of its elements satisfy the triangle inequality. L is autometrized if $S=L$, then regular if $d(a, O)=a$ for any a in L. A subgeometry of an autometrized lattice is a subset with the property that the distance for each pair of its points is an element of the subset. A Brouwerian algebra is defined as a lattice L with a least element O and a greatest element I such that with each two elements a, b there is associated a smallest element x such that $b+x>a$. This element is denoted by $a-b$. Then L autometrized by the symmetric difference $d(a, b)=(a-b)+(b-a)$ is a Brouwerian geometry. Summary. §2. Some aspects of Boolean geometries (Ellis' autometrized Boolean algebras) are extended to Brouwerian geometries, which leads to numerous characterizations of the former among the latter ones, e.g., absence of isosceles triangles, group property for the symmetric difference. Subgeometries of Brouwerian geometries are also considered. §3. Some general theorems on L-metrized spaces are established and the implications of the coincidence of lattice and metric betweenness in autometrized spaces are examined. We quote: A Brouwerian geometry is ptolemaic. An autometrized lattice with an I is a Boolean geometry if and only if it is regular, and metric and lattice betweenness coincide. §4. There are given a few results concerning the concept of congruence order together with some indications of possible future investigations. *Chr. Pauc* (Nantes).

Convex Domains, Extremal Problems, Integral Geometry

✶**Santaló, L. A. Introduction to integral geometry.** Actualités Sci. Ind., no. 1198 = Publ. Inst. Math. Univ. Nancago II. Hermann et Cie, Paris, 1953. 127 pp. 1500 francs.

The book consists of three parts. The first deals with the Euclidean plane and treats the following topics: density and measure for sets of points (it is emphasized from the outset that the densities are to be considered as obtained by exterior, rather than ordinary, multiplication of differentials). Density and measure for sets of straight lines, sets of pairs of points, sets of pairs of straight lines, kinematic measure, sets of segments, sets of rectifiable curves, fundamental formulas of Blaschke, applications (for instance, the isoperimetric inequality), lattices of figures. The author remarks in the introduction regarding the concepts of this first part: "However elementary and simple they may appear, we think it worthwhile to keep them always in mind to achieve a better understanding of the latest generalizations." This part is obviously most useful (besides being pretty), and it is merely a sad reflection on an at present very prevalent way of writing, that the author feels obliged to justify his procedure.

The second part deals with the geometry on surfaces; its subheadings are: density for sets of geodesics, geodesics which intersect a given curve, kinematic density on surfaces, integral geometry on surfaces of constant curvature. The author implies that, for instance, density for geodesics is (except for surfaces with constant curvature) not as well justified as for the plane, because there are no motions. Actually, the standard procedure in differential geometry of considering local concepts (like angle) as given by the local Euclidean geometry imposes this density.

The third part deals with general integral geometry and treats the topics: properties of Lie groups (and any others, the moving frames and relative components of E. Cartan, exterior differentiation, Pfaffian forms), density and measure in homogeneous spaces. This general theory is then applied to the following special cases: the group of central affinities in the plane, the unimodular group in the plane, the projective group, the generalized Poincaré formula in the plane, integral geometry in the plane of Cayley, the group of motions in n-dimensional Euclidean space.

There is a bibliography at the end of each major topic. Considering that there are only 123 pages of text, very much material is covered; nevertheless the book is unusually readable, and will serve its purpose very well.

H. Busemann (Copenhagen).

Santaló, Luis Antonio. Algebraic curves and analytic curves. Univ. Nac. Eva Peron Publ. Fac. Ci. Fisicomat. Serie Segunda. Revista 4, 493–506 (1953). (Spanish)

In der projektiven Ebene mit den homogenen Koordinaten x_0, x_1, x_2 wird die Normung

$$(1) \qquad (x\bar{x}) = x_0\bar{x}_0 + x_1\bar{x}_1 + x_2\bar{x}_2 = 1$$

eingeführt und die Gruppe Γ_q der linearen Substitutionen betrachtet, die (1) erhalten. Für ein Stück einer algebraischen Kurve wird dann nach E. Cartan der "Grad" eingeführt durch

$$(2) \qquad G = \frac{1}{2\pi i}\int dG, \quad dG = [dx_0 d\bar{x}_0] + [dx_1 d\bar{x}_1] + [dx_2 d\bar{x}_2].$$

Ist dann

$$(3) \qquad x_j^* = \sum a_j^k x_k$$

eine Transformation aus Γ_q, so werden die Pfaffschen Formen gebildet

$$(4) \qquad \omega_{jh} = \sum \bar{a}_h^i da_h^k$$

und ihr alternierendes Produkt

$$(5) \qquad \Omega = \prod \omega_{jh}$$

als Volumelement in der Gruppe Γ_q. Ist dann u ein Element aus Γ_q und C_1, C_2 Stücke analytischer Kurven, so wird der Satz bewiesen, dass die mittlere Schnittpunktzahl (definiert mit Ω) der ruhenden Kurve C_1 und der "bewegten" uG_2 (u aus Γ_q) gleich dem Produkt der Grade G_1, G_2 ist. Über verwandte Sätze vergleiche etwa die oben referierte Arbeit.

W. Blaschke (Istanbul).

Vidal Abascal, E. On the foundations of integral geometry. Mem. Real Acad. Ci. Madrid. Ser. Ci. Exact. 4, no. 4, 29 pp. (1953). (Spanish)

The paper contains interesting ideas which are however formulated in such a vague manner that reviewing becomes very difficult. The guiding principle is this: Consider a transitive Lie group G of transformations of an n-dimensional Riemann space R. If V^p and V^{n-p} are subvarieties of R of dimensions p and $n-p$, and V_α^{n-p} is the transform of V^{n-p} under the element α of G, then

$$\int_G (V^p, V_\alpha^{n-p}) d\alpha = \int_{V^p} \omega(V^{n-p}),$$

where (V^p, V_α^{n-p}) is the Kronecker index of V^p and V_α^{n-p} and $\omega(V^{n-p})$ is a suitable exterior differential form depending on V^{n-p}.

The applications are to surfaces of constant curvature with G as group of their motions. Let C be a simple closed curve. On either geodesic semi-tangent issuing from a variable point p of C lay off geodesic segments of (variable) lengths. These segments form a one-sided surface S with two, partly superimposed, sheets and a boundary curve C'. Denote by α the angle formed by a semi-tangent of C with C' at the corresponding point. If s' denotes arc length along C', then $\int_{C'} \cos \alpha ds'$ equals 0 for $C' \sim 0$ and equals $2L = 2\times$length C for $C' \sim C$. With the same notation, if $C' \sim C$, then $2L = \int_{G\cap C' \neq 0} (C', G) dG$ where G traverses the geodesics on S and dG corresponds to the standard density for geodesics.

The last part discusses a sort of duality between the Riemannian metrics

$$ds^2 = du^2 + G dv^2 \quad \text{and} \quad ds^2 = du^2 + (\partial\sqrt{G}/\partial u)^2 dv^2,$$

which corresponds to the special case where the general principle stated in the beginning becomes the Gauss-Bonnet theorem.

H. Busemann (Copenhagen).

Gołąb, S. On Finsler's measurement of an angle. Ann. Soc. Polon. Math. 24 (1951), no. 2, 78–84 (1954).

Let C be an oriented closed convex curve in the plane and p a point in the interior of C. There is a positive ϵ such that for any two points q_1, q_2 on C for which the angle q_1pq_2 is less than ϵ and q_2 follows q_1 the right-hand tangent of C at q_2 intersects the ray from p through q_1 beyond or at q_1. Here the converse is proved: If a closed curve C possesses everywhere a right-hand tangent and has the above property, then it is convex.

H. Busemann.

Aleksandrov, A. D., and Strel'cov, V. V. Estimates of the length of a curve on a surface. Doklady Akad. Nauk SSSR (N.S.) 93, 221–224 (1953). (Russian)

Let τ_r^+, τ_l^+ be the positive parts of right and left total geodesic curvatures of a curve L lying on a surface G homeomorphic to a disc, moreover σ_L^+ the positive part of the integral curvature of L as point set. Then $\sigma = \tau_r^+ + \tau_l^+ - \omega_L^+$ is a sort of excess for L. Denote by ω^+, ω^- the positive and regular parts of the integral curvature of G, and by p and d the perimeter and diameter of G. If $\omega^+ < 2\pi$, then the length s of L is bounded by a quantity which depends only on p, σ, and ω^+ (no such estimate exists for $\omega^+ \geq 2\pi$). If $\omega_0 = \omega^+ + \sigma \leq \pi$ then $s \leq p/(1+\cos \frac{1}{2}\omega_0)$; if $\pi < \omega_0 < 2\pi$, then $s \leq p/\sin \frac{1}{2}\omega_0$. If $\omega_0 < \pi$ and r is the distance between the endpoints of L, then $s \leq r/\cos \frac{1}{2}\omega_0$. In both these theorems the exact conditions for the equality sign are given. If G is convex (i.e., the boundary curve has non-negative geodesic curvature), then $p \leq (\pi + \frac{1}{2}\omega^-)d$ and if also $\pi \leq \omega_0 < 2\pi$ then $s < (\pi + \frac{1}{2}\omega^-)d/\sin \frac{1}{2}\omega_0$. If G is not convex then, in general, only $s < 3\pi d/2 \sin \frac{1}{2}\omega_0$.

H. Busemann.

Massera, J. L., and Schäffer, J. J. On the level curves of a convex surface. Bol. Fac. Ingen. Montevideo 4, 665–668 (1953). (Spanish)

In connection with investigations into the problem of whether or not a nested one-parameter family of convex

curves is the set of level curves of a convex function, the authors consider the following problem: given a family F of non-decreasing functions $f(t)$ on the interval $0 \leqq t \leqq 1$, does there exist a change of variable $t = T(s)$, monotone, $T(0) = 0$, $T(1) = 1$, such that $f(T(s))$ is convex for all f in F? Toward solving this problem, they define the negative variation $V^-(F, a, b)$ of a family F over an interval $[a, b]$ in the same manner as the negative variation of a function is defined, except that the supremum is taken simultaneously over all f in F as well as over all subdivisions of $[a, b]$. Here L is defined to be the family of all $f'(t)$, f in F. The following theorem is proved: If F contains the function $f(t) \equiv t$, a necessary and sufficient condition for the function $T(s)$ as described above to exist is that (a) all the functions of F are absolutely continuous, (b) if $[t', t'']$ is an interval completely interior to $[0, 1]$, $V^-(L, t', t'') < \infty$ (V^- is taken to be positive), (c) $\int_0^1 \exp[-V^{-1}(L, \tau, t)]dt < \infty$.

J. W. Green (Los Angeles, Calif.).

Avakumović, Vojislav G. Über die Scheitel der geschlossenen Kurven. Srpska Akad. Nauka. Zbornik Radova 35, Mat. Inst. 3, 147–152 (1953). (Serbo-Croatian. German summary)

A curve C in three-space is called convex if to any two points of C there exists a plane which has these and only these two points in common with C. Consider C as lying on a surface S. Then C has a geodesic curvature K_g and normal curvature K_n. The author proves that if C is convex and closed (and four times continuously differentiable) then both K_g and K_n have at least four extrema. When S is a sphere, this theorem includes Fog's generalization of a theorem of Mukhopadyaya [Fog, S.-B. Preuss. Akad. Wiss. 1933, 251–254]. *W. Feller* (Princeton, N. J.).

Fort, M. K., Jr. A cylindrical curve with maximum length and maximum height. Quart. J. Math., Oxford Ser. (2) 4, 314–320 (1953).

The reviewer recently considered [Amer. Math. Monthly 60, 30–31 (1953); these Rev. 14, 685] curves C which encircle a right circular cylinder of unit radius and which, regarded as point sets, have diameters equal to 2. Such a curve is rectifiable and its length is denoted by $L(C)$. Its height $H(C)$ is the minimum distance between two planes normal to the cylinder and containing C between them. It was proved that $2\pi \leqq L(C) \leqq 2^{3/2}\pi$ and $0 \leqq H(C) \leqq 2^{1/2}$ and that these bounds are best possible; however, the question was raised as to whether or not there exists a C for which $L(C) = 2^{3/2}\pi$ and also as to the best upper bound of $L(C) + H(C)$. In the present paper the author answers both questions by constructing a curve C for which $L(C) = 2^{3/2}\pi$ and $H(C) = 2^{1/2}$. The construction depends on defining a continuous function $f(\theta)$ on $[0, \frac{1}{2}\pi]$ with the properties (a) $f(0) = 0$ and $f(\frac{1}{2}\pi) = 2^{1/2}$, (b) $|f(\theta) - f(\phi)| \leqq 2 \sin \frac{1}{2}(\theta - \phi)$ for all θ, ϕ in the interval, and (c) the graph of f has length $\pi/2^{1/2}$. The construction of f is made by taking the limit of functions which are ingeniously made out of portions of the graph of the function $2 \sin \frac{1}{2}\theta$. The principal technical point is the proof that these "zigzag" functions satisfy (b), which is carried out by an intricate induction. *J. W. Green.*

Ohmann, D. Ungleichungen zwischen den Quermassintegralen beschränkter Punktmengen. II. Math. Ann. 127, 1–7 (1954).

With the notation used in the review to Part I [Math. Ann. 124, 265–276 (1952); these Rev. 13, 864], another special case, viz. $p = n - 1$, of the convex-figure inequality $(w_p)^{n-r} \geqq (v_n)^{p-r}(w_r)^{n-p}$ $(0 \leqq r < p \leqq n)$, is now extended to point-sets, which are, however, now restricted to be bounded and closed. *L. C. Young* (Madison, Wis.).

Algebraic Geometry

Arvesen, Ole Peder. Sur les paraboles, considérées comme des courbes Γ'. Norske Vid. Selsk. Forh., Trondheim 26 (1953), 85–88 (1954).

L'auteur considère des cas particuliers de certaines courbes étudiées dans une note précédente [même Forh. 12, 115–118 (1940); ces Rev. 2, 13]. *M. Piazzolla Beloch.*

Rosina, B. A. Ulteriori osservazioni sulle coniche generalizzate. Ann. Univ. Ferrara. Sez. VII. (N.S.) 2, 117–127 (1953).

Generalized conics were defined by Piazzolla-Beloch [see Ann. Univ. Ferrara. Parte I. 6, 91–101 (1947); these Rev. 13, 155]. The generalized conics C^{2n} treated in the present paper have an n-fold contact with the line at infinity at each of two simple points of C^{2n}. The author shows that the C^{2n} of elliptic or hyperbolic types are the only algebraic plane curves that have two and only two principal diameters perpendicular to each other, and that those of parabolic type are the only plane curves of even order with one and only one principal diameter. He also shows that the C^{2n} are the only algebraic curves which have an infinite number of pairs of mutually conjugate diameters.

T. R. Hollcroft (Aurora, N. Y.).

Hutcherson, W. R., and Gormsen, S. T. Maps of certain algebraic curves invariant under cyclic involutions of periods three, five, and seven. Canadian J. Math. 6, 92–98 (1954).

This paper is a study of the projective model F of a linear system of plane curves of order seven and dimension five and such that each curve of the system is invariant under a cyclic collineation of period seven. The authors use the methods of classical algebraic geometry to determine the order and singular points of F and to study certain linear systems of twisted septimic curves on F. *G. B. Huff.*

Salmon, Paolo. Sulla postulazione di una curva semplice dello spazio S_r. Boll. Un. Mat. Ital. (3) 9, 46–50 (1954).

Let Θ_m be the postulation of an algebraic (possibly reducible) curve C^n in S_r, of order n, on the hypersurfaces of order m constrained to contain C^n. The author gives another proof of the well known result (a special case of the Hilbert postulation formula) that $\Theta_m - mn$ is a constant if m is large. He seems to believe that his proof is more elementary than the customary proof of that theorem. *O. Zariski.*

Gherardelli, Francesco. Sul gruppo della torsione delle varietà abeliane di rango due. Univ. e Politecnico Torino. Rend. Sem. Mat. 12, 293–300 (1953).

The author determines the torsion group of an abelian variety of rank 2, image of the involution generated by the transformation $u \rightarrow -u$ on a Picard variety. If p is the genus of the associated Riemann matrix, the order of the group is in general 2^{2p+1}; the exceptional cases being (a) that in which the divisors of the Riemann matrix are all unity,

when the order of the group is 2^{2p}, and (b) that in which γ of the divisors are equal to 1 and the remaining $p-\gamma$ to 2 ($\gamma>0$), when the order is $2^{2p+1}-2^{2(p-\gamma)}$. *J. A. Todd.*

Andreotti, Aldo. Sopra il gruppo della torsione unidimensionale delle varietà algebriche. Univ. e Politecnico Torino. Rend. Sem. Mat. 12, 239–264 (1953).

After some introductory remarks about the fundamental group of an algebraic variety, the author establishes the following theorem. Let V_d be a non-singular algebraic variety, of irregularity q, and let $\theta_1(V_d)$ be the one-dimensional torsion group of V_d. Let $\theta_1(W_d)$ be any subgroup of $\theta_1(V_d)$. Then there exists an algebraic variety W_d, having the same Picard variety as V_d, with one-dimensional torsion group $\theta_1(W_d)$, carrying an involution of order equal to the index of $\theta_1(W_d)$ in $\theta_1(V_d)$, generated by an abelian group of birational transformations isomorphic to the factor group $\theta_1(V_d)/\theta_1(W_d)$, and representing W_d rationally on V_d.

Next the author proceeds to calculate the torsion group of some special varieties (sextic surface of Enriques, Kummer surface, Wirtinger variety). He next shows how to construct a variety of given dimension, irregularity, and torsion group. The method is to construct first a regular surface with cyclic torsion group of given order. By forming the product variety of several such surfaces and a Picard variety of dimension q, a variety U is obtained, with irregularity q and any given torsion group. A general surface section of $\bar{U}$ has the same irregularity and torsion group, and by forming the product with a linear space of suitable dimension varieties of any assigned dimension and the same property can be obtained.

The paper concludes with a determination of the order of the torsion group of elliptic surfaces with an elliptic pencil of curves of genus three, recently studied by Conforto and Gherardelli [Ann. Mat. Pura. Appl. (4) 33, 273–351 (1952); these Rev. 14, 681]. *J. A. Todd.*

Chisini, Oscar. Un caratteristico procedimento dimostrativo della geometria algebrica. Univ. e Politecnico Torino. Rend. Sem. Mat. 12, 21–36 (1953).

This paper is an account of a lecture by the author, in which a variety of known geometrical theorems are proved using the analytical theorem that a single-valued algebraic function is rational, and, if it has no poles, reduces to a constant.

J. A. Todd (Cambridge, England).

Differential Geometry

***Marussi, Antonio. Principi di Geodesia intrinseca applicati al campo di Somigliana.** Società Italiana per il Progresso delle Scienze, XLII riunione, Roma, 1949, Relazioni, Vol. primo, pp. 247–255. Società Italiana per il Progresso delle Scienze, Roma, 1951.

The fundamentals of the differential geometry of the earth's gravitational potential field are reviewed with particular reference to the gradients of the principle curvatures on the earth's surface. Results are applied to a specific formulation of the spheroid and the corresponding gravitational acceleration.

N. A. Hall (Minneapolis, Minn.).

Ara, Rahmat, und Pinl, M. Zur integrallosen Darstellung reeller isotroper Kurven. J. Reine Angew. Math. 192, 204–209 (1953).

The quadratic form

$$ds^2=dx_1^2+dx_2^2+\cdots+dx_k^2-dx^2_{k+1}-\cdots-dx_n^2$$

defines the arc length in pseudo-euclidean n-space. The number $\alpha=n-k$ is called the index of inertia of the form. If $0<\alpha<n$ there are real isotropic curves in the space. The authors here consider integral-free representations of the real isotropic curves for the following cases: $n=3$, $\alpha=1$; $n=4$, $\alpha=1$; $n=4$, $\alpha=2$. *S. B. Jackson.*

Franckx, Ed. Sur la théorie des courbes qui appartiennent à une surface et y ont un contact d'ordre k. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 125–129 (1954).

[For part I see same Bull. (5) 39, 629–635 (1953); these Rev. 15, 251.] Let $E^{(1)}$ represent a set of curves having a common tangent at a point M in E_3. The curves of $E^{(2)}$, a subset having contact of order 2, must have a common (a) osculating plane, and (b) radius of curvature. Meusnier's Theorem says that if in $E^{(2)}$ one takes $E_s^{(2)}$, the subset of curves which belong to a given surface, then condition (a) implies condition (b) (except for the asymptotic directions). The curves of a subset $E^{(k+1)}$ chosen in a set $E^{(k)}$ must have common values for (a) $d^{k-1}\rho/ds^{k-1}$ and (b) $d^{k-2}\tau/ds^{k-2}$ at M, where ρ and τ are the curvature and torsion. If one takes the subset $E_s^{(k)}$ of those curves of $E^{(k)}$ which belong to a given surface, do the two conditions reduce to one? The answer is yes. For a set $E_s^{(k)}$, $k\geqq 2$, either one of $d^{k-1}\rho/ds^{k-1}$ or $d^{k-2}\tau/ds^{k-2}$ is a local invariant or there is a linear relationship between them of the form

$$\cos\theta d^{k-1}\rho/ds^{k-1}+\rho\sin\theta d^{k-2}\tau/ds^{k-2}=c_k,$$

where θ is the angle between the common principal normal and the normal to the surface and c_k is a constant for the $E_s^{(k)}$. *A. Schwartz* (New York, N. Y.).

Franckx, Ed. Sur la théorie des courbes qui appartiennent à une surface et y ont un contact d'ordre k. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 262–265 (1954).

Consider $E_s^{(k)}$, a set of curves on a given surface having contact of k at a common point M. Let the arc-length parameter be chosen so that for each curve one obtains the point M by choosing $s=s_0$. Then for the normal curvature $\rho_n=\rho_n(s)$ and the geodesic torsion $\tau_g=\tau_g(s)$ (interpreted as plane curves) we have the following generalization of the classical theory of Meusnier and Bonnet: All the curves $\rho_n=\rho_n(s)$ for a set $E_s^{(k)}$ have themselves contact of order $k-1$ at $s=s_0$. The same is true for the curves $\tau_g=\tau_g(s)$. *A. Schwartz* (New York, N. Y.).

Henderson, G. P. Parallel curves. Canadian J. Math. 6, 99–107 (1954).

Two curves of class C^{n+1} in E^n with nonvanishing curvature of all orders are called parallel if there is a 1-1 correspondence between their points such that at corresponding points the tangents are parallel and the line connecting the points is perpendicular to these tangents. Denote by F_p a p-dimensional ($p=1, \cdots, n-1$) family of mutually parallel curves. The curve C is a pth involute of D (and D a pth evolute of C) if C is an orthogonal trajectory of the oscillating p-flats of D. The pth involutes of a curve form an F_p. Unless the curves of an F_p be on concentric hyperspheres they have exactly one common pth evolute. There are

several theorems on the so-called pth polar developable of an F_{n-1} whose definition is too involved to be included here.
H. Busemann (Copenhagen).

Scherk, Peter. Intorno alle curve sferiche. Boll. Un. Mat. Ital. (3) **9**, 38–40 (1954).

The author, in a letter to B. Segre, establishes in a simple way a theorem of Gallarati [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 238–241 (1952); these Rev. **15**, 155] which states that a necessary and sufficient condition that a surface of class C''' be a sphere is that for every curve on it of class C''' the differential of arc on the binormal indicatrix is a constant multiple of the differential of arc on the locus of the center of curvature. He also obtains results of Segre and Scherrer concerning the integrals of certain functions of the curvature and torsion for closed spherical curves. It may be noted that the theorem that the integral of the torsion around a closed spherical curve vanishes, which is attributed to Scherrer, was published by Fenchel [Tôhoku Math. J. **39**, 95–97 (1934)]. *S. B. Jackson.*

Kostyučenko, A. G. On a connection between the structure of an $(n-1)$-dimensional surface and its principal curvatures. Uspehi Matem. Nauk (N.S.) **8**, no. 5(57), 161–164 (1953). (Russian)

The main results are: If on a hypersurface in E^n one of the principal curvatures has the constant value $\lambda>0$, then the surface is the envelope of an $(n-2)$-parameter family of spheres with radius λ^{-1}; for $\lambda=0$ the spheres have to be replaced by planes. There is no surface on which two of the principal curvatures have constant values which are different from each other and from 0 (the case where one of two constant principal curvatures vanishes occurs on cylindrical surfaces). If at every point of the hypersurface p of the principal curvatures coincide, then every point belongs to a piece of a p-dimensional sphere lying on the surface.
H. Busemann (Copenhagen).

Vidal Abascal, E., and Rodeja F., E. G. Note on curves on surfaces of constant curvature. Collectanea Math. **5**, 331–337 (1952). (Spanish)

On a surface with constant curvature consider a simple closed curve C and variable geodesic triangle, whose sides have constant lengths and such that two of its vertices slide along C. The paper gives an explicit formula for the area bounded by the curve traversed by the third vertex, which is too involved for reproduction here. It is shown that various known results are contained in this formula.
H. Busemann (Copenhagen).

Efimov, N. V. Investigation of a complete surface of negative curvature. Doklady Akad. Nauk SSSR (N.S.) **93**, 393–395 (1953). (Russian)

Efimov, N. V. Investigation of a single-valued projection of a surface of negative curvature. Doklady Akad. Nauk SSSR (N.S.) **93**, 609–611 (1953). (Russian)

The first paper proves: A surface $z=f(x, y)$ which is defined and regular for all x, y cannot have a Gauss curvature which stays below a fixed negative constant. The example $z=e^x \sin y$ shows that a surface may have this property in a strip between parallel lines. The second paper establishes the theorem: If $z=f(x, y)$ is regular in a square of the (x, y)-plane and has curvature $\leqq -1$, then the sides of the square have at most length 14. The exact value of the greatest lower bound for the side length is not known.
H. Busemann (Copenhagen).

Rembs, Eduard. Verbiegbarkeit konvexer Kalotten. Math. Ann. **127**, 251–254 (1954).

Given a convex surface homeomorphic to a disc. Its tangent planes along its boundary curve are supposed to pass through a given point. Then this surface is rigid under infinitesimal isometric deformations which preserve this property of the boundary. *P. Scherk.*

Goebel, Wolfgang. Biegungsflächen der Rotationsellipsoide mit konischen Punkten. Math. Nachr. **11**, 5–34 (1954).

This paper contains new bending surfaces of the ellipsoid of revolution, i.e., surfaces isometric to ellipsoids of revolution. Among these are some for which the surface is closed and which are regular except for two conical points which correspond, in the isometry, to the endpoints of the diameter of revolution of the ellipsoid. The equations of the surfaces are derived from certain Enneper surfaces by use of the Bäcklund transformation. *S. B. Jackson.*

***Heinz, Erhard. Sur les solutions de l'équation de surface minimum.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 61–65. Centre National de la Recherche Scientifique, Paris, 1953.

Let Σ denote a minimal surface given in non-parametric form $z=z(x, y)$, over $(x-x_0)^2+(y-y_0)^2<R^2$. If $K(x, y)$ is the Gaussian curvature, then the author shows:

$$|K(x_0, y_0)| \equiv |K_0| \leqq 3\pi^2/R^2.$$

As $R\to\infty$, we obtain Bernstein's classic result that Σ must be a plane. The author also obtains a related inequality for C'' solutions of the equation $a\varphi_{xx}+2b\varphi_{xy}+c\varphi_{yy}=0$, where $ac-b^2\equiv1$, $a_x+b_y\equiv b_x+c_y\equiv0$. The proofs depend upon a result due to H. Lewy [Bull. Amer. Math. Soc. **42**, 689–692 (1936)]. *M. O. Reade* (Ann Arbor, Mich.).

Goldberg, Michael. Rotors within rotors. Amer. Math. Monthly **61**, 166–171 (1954).

Let C_1 be the involute of the deltoid (=the hypocycloid of three cusps), C_2 the curve of Ribaucour (=a parallel to the astroid). C_1 is shown to be a rotor in a square, C_2 to be a rotor in an equilateral triangle. The author proves the theorem: C_2 can be inscribed in C_1 and may be turned continuously in a constrained manner through all orientations. It is not a special case of a similar known theorem stating that a hypocycloid of n cusps is a rotor in another one of $n+1$ cusps. *O. Bottema* (Delft).

Gambier, B. Epi- ou hypocycloïdes tangentes à 3 droites. J. Math. Pures Appl. (9) **33**, 1–28 (1954).

Consider any family (Γ) of closed epicycloids or hypocycloids which are similar to each other. The common tangents to two such curves consist of n^2 tangents where n is the class of the curves. These n^2 tangents can be divided into n classes of n each. The n tangents of any one class are tangent to a conic. It is shown that the members of (Γ) tangent to three given lines may be divided into n^2 one-parameter families and that the curves of any one family have collinear centers and have actually n common tangents which are also tangent to a conic. The number of curves of family (Γ) which are tangent to 4 arbitrary lines is found to be n^3. *S. B. Jackson* (College Park, Md.).

Elianu, I. P. Réseaux multiples. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **3** (1951), 457–466 (1952). (Romanian. Russian and French summaries)

L'auteur considère p surfaces S^α $(\alpha=1, 2, \cdots, p)$ de l'espace projèctif E_{n-1} rapportées aux mêmes coordonnées curvilignes (u, v), et les variétés linéaires suivantes: $U_p{}^\alpha=\{x^1, \cdots, x^p, \partial x^\alpha/\partial u\}$, $V_p{}^\alpha=\{x^1, \cdots, x^p, \partial x^\alpha/\partial v\}$ contenant les points homologues x^i et tangentes respectivement aux courbes u et v de S^α. Il suppose que les $V_p{}^\alpha$ contiennent des points $x^\alpha{}_{(1)}$ [ou les $U_p{}^\alpha$ des points $x^\alpha{}_{(-1)}$], qui, pour $v=v_0$ [ou $u=u_0$] décrivent des courbes respectivement tangentes à l'une des variétés linéaires

$$W^{(1)}_{2p-1}=\{x^1, \cdots, x^p, x^1_{(1)}, \cdots, x^p_{(1)}\}, \quad W^{(1)}_{p-1}=\{x^1_{(1)}, \cdots, x^p_{(1)}\}$$

ou

$$W^{(-1)}_{2p-1}=\{x^1, \cdots, x^p, x^1_{(-1)}, \cdots, x^p_{(-1)}\},$$
$$W^{(-1)}_{p-1}=\{x^1_{(-1)}, \cdots, x^p_{(-1)}\},$$

et dit alors que les courbes (u, v) forment sur S^α un réseau multiple d'ordre p. Il montre que dans ces conditions les coordonnées du point courant de S^α satisfont un système d'equations aux que dèrivées partielles que, vu sa forme matricielle, il appelle du type de Laplace, introduit des invariants matriciels, et associe au réseau multiple de départ une suite (de Laplace) de réseaux multiples. Il étudie spécialment le cas d'un réseau multiple décomposable en réseaux simples, et celui où le réseau envisagé a des invariants matriciels scalaires. *P. Vincensini* (Marseille).

Marcus, F. Sur les surfaces et réseaux $\mathcal{S}$ et sur les surfaces de Ionas. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 519–526 (1952). (Romanian. Russian and French summaries)

Dans cet article l'auteur étudie les surfaces $\mathcal{S}$, portant des réseaux conjugués (x), tels que les invariants h et k de leur équation de Laplace ponctuelle soient égaux, respectivement, aux invariants h et k de leur équation tangentielle (réseaux $\mathcal{S}$). Il établit que si un réseau (x), et l'un de ses transformés de Laplace (x_1) ou (x_{-1}) dans le sens des lignes v ou u est $(\mathcal{S})$, la suite de Laplace du réseau (x) est autoprojective, et l'on a $hk=1$, $(\log h)_{uv}=2(h-1/h)$. Réciproquement, s'il y'a projectivité entre un réseau $\mathcal{S}$ et l'un de ses deuxièmes transformés de Laplace, (x_1) et (x_{-1}) sont du type $(\mathcal{S})$, et l'on retrouve ainsi un résultat caractérisant les surfaces de Jonas comme supports de réseaux dont les deux congruences focales sont de Waelsh. *P. Vincensini.*

Arghiriade, Em. Sur les surfaces de Čech. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 85–99 (1952). (Romanian. Russian and French summaries)

L'auteur étudie les surfaces (de Čech) à lignes de Darboux planes, surfaces qui peuvent être réparties en trois classes (S_1, S_2, S_3). En ce qui concerne les surfaces S_1, Čech a montré que les plans de leurs lignes de Darboux enveloppent trois cônes à sommets (O_1, O_2, O_3) colinéaires, et il a dit, par inadvertance, que ces cônes étaient quadratiques. En outre les trois cônes sont en perspective, les plans des perspectivités passant par une même droite coupant la droite $O_1O_2O_3$ en trois points déterminant le covariant cubique des trois sommets. L'auteur relève l'inadvertance signalée, et montre que les cônes en question ne peuvent jamais être quadratiques.

Au sujet des mêmes surfaces S_1, il démontre que les cônes circonscrits à une S_1 ayant pour sommets les trois points O_1, O_2, O_3, touchent la surface suivant des courbes planes dont les plans coïncident avec ceux des perspectivités de Čech. Pour les surfaces S_2 il établit une propriété nouvelle relative à la coïncidence de deux des cônes de Čech. Pour les surfaces S_3, pour lesquelles Čech a montré que les plans des lignes de Darboux (de même que ceux des lignes de Segre) appartiennent à trois faisceaux, l'auteur établit que ces surfaces sont des surfaces de coïncidence, les droites canoniques passant par un point fixe P non situé dans le plan des points $O_1O_2O_3$ (qui ne sont plus alignés), plan des axes des faisceaux des plans des lignes de Darboux. Les axes des trois faisceaux des plans des lignes de Segre sont, pour ces mêmes surfaces S_3, les droites PO_1, PO_2, PO_3. *P. Vincensini* (Marseille).

Demaria, Davide Carlo. Invarianti affini di elementi curvilinei. Boll. Un. Mat. Ital. (3) **9**, 40–45 (1954).

First, the author gives four affine invariants of a pair of plane E_2 which do not have common tangent lines, using these tangent lines as the axes of a reference system. Second, he gives the affine invariants for a pair of E_2 in space. The intersection of the two osculating planes (assumed not parallel) is taken as the z-axis; the origin is placed at the midpoint of the segment intercepted on the z-axis by the tangent lines; the x and y axes are taken parallel to the tangent lines. Third, he gives the affine invariants for a trio of E_2 in space. If P_i and t_i, $i=1, 2, 3$, are the centers and tangents for the three elements, the coordinate planes of a Cartesian reference frame are taken to be the planes determined by P_1t_2, P_2t_3, P_3t_1. All the invariants are given simple affine characterizations. The work of the first part covers cases other than those studied by L. A. Santaló [Duke Math. J. **14**, 559–574 (1947); these Rev. **9**, 201]; the work of the third part uses a different reference frame from that of R. Cherep [Gaz. Mat., Lisboa **12**, no. 50, 35–38 (1951); these Rev. **13**, 773], who did not arrive at a complete set of invariants. *A. Schwartz.*

Salini, Ugo. Calotte superficiali del terzo ordine inflessionali con lo stesso centro e lo stesso piano tangente. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11** (1952), 434–452 (1953).

In relation with earlier papers of E. Bompiani [same Rend. (5) **2**, 261–291 (1941); these Rev. **8**, 349] and C. Longo [ibid. **7**, 295–326 (1948); these Rev. **10**, 570] the set of all inflexional caps of the third order (in ordinary projective space) with the same origin and the same tangent plane is studied with respect to the projective group. These caps can be considered as points of a projective four. dimensional space with respect to the group of collineations leaving invariant: a three-dimensional cone of the fourth order; a two-dimensional cone of the third order belonging to the preceding one and with the same vertex; a hyperplane (not through the vertex). Linear systems of such caps and their invariants are examined. *E. Bompiani.*

Vaona, Guido. Le trasformazioni fra due spazi che posseggono iperpiani di rette caratteristiche. Univ. e Politecnico Torino. Rend. Sem. Mat. **12**, 195–238 (1953).

L'auteur étudie les transformations entre S_r projectifs qui possèdent, en chaque point générique, ∞^h $(1\leq h\leq r-2)$ droites caractéristiques. Pour $h=r-2$, on a les deux cas: (a) de chaque point générique sont issues ∞^{r-2} droites charactéristiques formant un cône quadratique, et une droite caractéristique isolée; (b) les droites caractéristiques issues de chaque point générique comprennent un hyperplan et, en général, r droites isolées. Les transformations du

type (a) ont été étudiées par E. Čech. L'article actuel est consacré à la détermination de toutes les transformations du type (b) pour lesquelles $r \geqq 4$. Après l'établissement de divers résultats de caractère local et une fusion opportune des nombreux cas projectivement distincts qui peuvent se présenter, l'auteur se ramène à la considération d'un type de transformation, étudié par Čech (possédant deux S_{r-1} de droites caractéristiques en chaque point), dont il prolonge certains résultats soit par des représentations analytiques soit par des constructions géométriques; puis d'un deuxième type, constitué par les transformations qui possèdent un hyperplan de droites caractéristiques et des S_h ($h<r-1$) de telles droites, et qu'il étudie en détail en se basant sur une représentation canonique particulièrement commode des homographies entre espaces superposés. Il montre que les calottes hyperplanes du premier ordre lieux de directions caractéristiques, peuvent être distribués suivant les éléments de ∞^1 hypersurfaces, et que ces ∞^1 hypersurfaces sont ou des hyperplans, ou des développables lieux de ∞^1 S_{r-2} admettant un S_{r-1} tangent fixe le long de chaque S_{r-2} générateur. Chacun des deux cas indiqués est étudié de façon approfondie, et caractérisé par des propriétés diverses et par d'intéressantes constructions géométriques.

P. Vincensini (Marseille).

Muracchini, Luigi. Trasformazioni puntuali fra due spazi che posseggono un'unica congruenza di curve caratteristiche. Univ. e Politecnico Torino. Rend. Sem. Mat. **12**, 159–176 (1953).

Dans une transformation entre deux espaces projectifs S_3, $\bar{S}_3$ il existe, en un couple régulier de points homologues A, $\bar{A}$ sept droites caractéristiques, génératrices de base d'un réseau de cônes cubiques (de sommets A où $\bar{A}$). Il peut arriver, et cela de quatre façons projectivement distinctes, que les sept droites caractéristiques coïncident. L'auteur envisage, parmi les types de transformations ponctuelles réalisant la coïncidence indiquée, ceux, au nombre de trois, pour lesquels il-y-a coïncidence des droites caractéristiques sur une nappe cuspidale (de second ordre) de cône. Il donne de ces transformations des constructions géométriques projectives, basées sur un ensemble de formules et de propriétés établies pour les transformations ponctuelles générales entre S_r projectifs, qui ont un interêt propre, et permettraient, par exemple, une étude locale, poussée jusqu'au voisinage du troisième ordre, de transformations jouissant de propriétés affectant le voisinage du premier ordre d'un couple quelconque de points homologues. La méthode d'investigation employée est celle du déplacement infinitésimal du repère mobile projectif de E. Cartan.

P. Vincensini (Marseille).

Muracchini, Luigi. Sulle trasformazioni puntuali inviluppi di omografie. Boll. Un. Mat. Ital. (3) **8**, 390–398 (1953).

Let T be a point-transformation between two projective spaces S_n, $\bar{S}_n$ and Φ a system of ∞^s homographies ($1 \leqq s \leqq n$) between the same spaces. If a family of ∞^s V_{n-s} in S_n exists such that $\bar{V}_{n-s} = TV_{n-s}$ is also the transform of V_{n-s} by an homography of Φ and if this homography is tangent to T at all pairs of corresponding points of V_{n-s}, $\bar{V}_{n-s}$, then T is said to be an envelope of Φ. The problem of determining such transformations T has been tackled and partially solved by E. Čech. The author gives some theorems pointing to the solution for arbitrary values of s and n. *E. Bompiani.*

Have

***Villa, M. Recherches de types particuliers de transformations ponctuelles.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 67–77. Centre National de la Recherche Scientifique, Paris, 1953.

Revue d'ensemble de résultats récents sur des types particuliers de transformations ponctuelles entre deux espaces projectifs. *E. Bompiani* (Rome).

Mastrogiacomo, Pasquale. Trasformazioni puntuali tra spazi proiettivi osculabili con trasformazioni quadratiche di terza specie particolari. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12** (1953), 285–298 (1954).

It is well known that a point-transformation between two projective spaces in the neighborhoods of two regular corresponding points cannot be osculated in general by quadratic transformations. The cases in which such osculation is possible have been studied by A. Cossu [same Rend. (5) **10**, 448–467 (1951); these Rev. **14**, 86]; the author examines the case of an osculating quadratic transformation of third species of particular types (some of the fundamental points become infinitely near). *E. Bompiani* (Rome).

Castoldi, Luigi. Appunti per una interpretazione geometrica del formalismo delle connessioni proiettive. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12** (1953), 426–439 (1954).

A transformation of asymmetric connections in a fundamental topological space X_n is called a transformation T_p if it preserves parallelism along any curve in X_n, and a transformation of symmetric connections in X_n is called a projective transformation if it preserves the paths in X_n. It is shown that there are certain transformations T_p of any asymmetric connection L in X_n such that the Thomas tensor of X_n with respect to L is the curvature tensor of X_n with respect to any transformed connection of L by any of those transformations T_p. Similarly, there are certain projective transformations of any symmetric connection Γ in X_n such that the Weyl tensor of X_n with respect to Γ is the tensor defined by

$$\overset{*}{\hat{R}}_{ijk}{}^{h} = \overset{*}{\hat{\Gamma}}_{ijk}{}^{h} + \frac{1}{n-1}\left(\delta_i^h \overset{*}{\hat{\Gamma}}_{jk} - \delta_j^h \overset{*}{\hat{\Gamma}}_{ik}\right),$$
$$\overset{*}{\hat{\Gamma}}_{ik} = \overset{*}{\hat{\Gamma}}_{ikh}{}^{h},$$

where $\overset{*}{\hat{\Gamma}}_{ijk}{}^{h}$ are the components of the curvature tensor of X_n with respect to any transformed connection of Γ by any of those projective transformations. *C. C. Hsiung.*

Kobayashi, Shôshichi. Groupe de transformations qui laissent invariante une connexion infinitésimale. C. R. Acad. Sci. Paris **238**, 644–645 (1954).

It is proved that in certain important cases the group of transformations which leaves invariant an infinitesimal connection in a differentiable fiber bundle is a Lie group. These cases include: (1) the group of affine transformations [cf. Nomizu, Proc. Amer. Math. Soc. **4**, 816–823 (1953); these Rev. **15**, 468]; (2) the group of projective transformations; (3) the group of conformal transformations.

S. Chern (Chicago, Ill.).

Guggenheimer, Heinrich. Vierdimensionale Einsteinräume. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11** (1952), 362–373 (1953).

The paper studies the properties of the characteristic classes of a compact complex manifold of two complex

dimensions, which has a complex Einstein metric. The characteristic classes c^2, c^4 are integral cohomology classes of dimensions 2 and 4, respectively. Paired with the fundamental homology class M of the manifold, one gets two integers

$$(c^2 \cup c^2)\cdot M = p^{(1)}-1, \quad c^4\cdot M=\chi,$$

of which χ is the Euler-Poincaré characteristic and $p^{(1)}$ generalizes the linear genus of an algebraic surface. It is proved that a necessary condition for a compact complex manifold of complex dimension 2 to carry an Einstein metric is that the inequality $0\leqq p^{(1)}-1\leqq 3\chi$ holds. Moreover, if $\chi=0$, the metric must be locally Euclidean. Various remarks are given on the properties of the Einstein metric and its characteristic classes, including a table of possible algebraic surfaces with an Einstein metric, for $\chi\leqq 8$.

Assuming that the Ricci tensor does not vanish identically, the author next studies homogenetic metrics in the sense of Bochner. A table is given of the possible homogenetic Einstein metrics on a compact four-dimensional Kähler manifold for which the components of the curvature tensor are integers.

In an appendix a generalization of the above notions to higher dimensions is suggested. Necessary conditions are obtained on the characteristic classes, in order that such a metric exists on the manifold. *S. Chern* (Chicago, Ill.).

Westlake, W. J. Hermitian spaces in geodesic correspondence. Proc. Amer. Math. Soc. **5**, 301–303 (1954).

Some time ago, the reviewer [Bull. Amer. Math. Soc. **47**, 901–910 (1941); these Rev. **3**, 191] derived the differential equations satisfied by a geodesic in a Hermitian space and stated that the necessary and sufficient condition for the correspondence between geodesics of two Hermitian spaces with symmetric connections (Kähler spaces) is that the connections coincide except for a special tensor. S. Bochner [ibid. **53**, 179–195 (1947); these Rev. **8**, 490] pointed out that this last result is incorrect. The present author shows that the special tensor, introduced by the reviewer, vanishes identically. Thus, the condition for geodesic correspondence is that the connections of the two Kähler spaces coincide. This is done by considering the complex conjugate equations of a geodesic together with the original equations. Finally, by use of similar methods, the author determines necessary and sufficient conditions for the correspondence of geodesics in two general Hermitian spaces. *N. Coburn* (Ann Arbor, Mich.).

Chern, Shiing-shen. Relations between Riemannian and Hermitian geometries. Duke Math. J. **20**, 575–587 (1953).

Eine Riemannsche Metrik in einer $2n$-dimensionalen Mannigfaltigkeit M heisst "lokal Hermitesch" ("lokal Kaehlersch"), wenn man in jeder hinreichend kleinen Umgebung komplexe Koordinaten $z^1, \cdots, z^n$ einführen kann, in welchen die Riemannsche Metrik Hermitische (Kaehlersche) Form annimmt; die lokalen komplexen Koordinatensysteme brauchen dabei nicht komplex-analytisch miteinander zusammenzuhängen. In der vorliegenden Arbeit werden Krümmungsbedingungen aufgestellt, welche notwendig und hinreichend dafür sind, dass eine gegebene Riemannsche Metrik lokal Hermitesch, bzw. lokal Kaehlersch ist: Es muss eine Familie orthogonaler Bezugssysteme geben (eine Teilmannigfaltigkeit des Hauptfaserraumes über M, welcher aus allen orthogonalen Bezugssystemen besteht), in denen die Komponenten des Krümmungstensors gewisse Symmetrieeigenschaften aufweisen. Diese sind ein Ausdruck der Integrabilitätsbedingungen für die äussern Differentialsysteme, deren Integration die gewünschten lokalen komplexen Koordinaten liefert. Die Bedingung für den lokal Kaehlerschen Fall wird auch mit Hilfe der gemischten Krümmung zweier Bivektoren formuliert. Es ergibt sich u.a., dass eine Riemannsche Metrik konstanter Krümmung K nur dann lokal Kaehlersch sein kann, wenn $K=0$ ist. *B. Eckmann* (Zürich).

Have

★Davies, E. T. Sur la théorie invariante des transformations de contact. Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 11–15. Centre National de la Recherche Scientifique, Paris, 1953.

The author shows that Finsler spaces and in general spaces in which the metric is based upon a function $L(x, u)$, u being a covariant or contravariant vector density, can be treated as nonholonomic subspaces of a Riemannian V_{2n} in which the transformations considered are contact transformations. It is supposed that the connecting quantities $(B_i{}^a, C_a{}^p)$ of the nonholonomic space satisfy the relations $B_i{}^j=\delta_i{}^j$ and $C_q{}^p=\delta_q{}^p$. Then $B_i{}^p$ and $C_j{}^p$ define two quantities B_{ij} and C^{ij}. The spaces mentioned above are obtained by certain restrictions on the geometry of V_{2n} and the quantities B_{ij} and C^{ij}. *J. Haantjes* (Leiden).

Nomizu, Katsumi. Sur l'algèbre d'holonomie d'un espace homogène riemannien. C. R. Acad. Sci. Paris **238**, 319–321 (1954).

Let G/H be a homogeneous space with Riemannian structure [cf. Nomizu, same C. R. **237**, 1386–1387 (1953); these Rev. **15**, 468]. The author shows that if the identity component $\tilde{H}_0$ of the isotropy group is irreducible, and if G/H is not locally euclidean, then the holonomy group of G/H is irreducible. Then follows a description of the structure of the Lie Algebra of the holonomy group, in which special attention is paid to the cases when G/H admits a Kähler structure, and when G is compact. *A. Nijenhuis* (Princeton, N. J.).

Kobayashi, Shôshichi. La connexion des variétés fibrées. I. C. R. Acad. Sci. Paris **238**, 318–319 (1954).

The author outlines an alternative treatment of the theory of connections in a fiber bundle by putting emphasis on the tangent bundle of the bundle space. The holonomy groups are defined, and it is stated that the restricted holonomy group is a Lie group. The paper also gives the theorem that in order that the structural group G of a bundle be reducible to a subgroup H, it is necessary and sufficient that a connection exists in the bundle whose holonomy group is contained in H. No proofs are given. *S. Chern.*

Kobayashi, Shôshichi. La connexion des variétés fibrées. II. C. R. Acad. Sci. Paris **238**, 443–444 (1954).

[Cf. the preceding review.] Let Ω_x be an H-space of piecewise differentiable paths at a point x of the base space B of a bundle. An equivalence relation can be defined in Ω_x and the quotient space $\tilde{\Omega}_x$ obtained is a topological group. By means of the holonomy group of a connection in the bundle, one defines a homomorphism of $\tilde{\Omega}_x$ into G. The following theorems are given. (1) In order that two bundles be equivalent it is necessary and sufficient that there are connections in both which give the same equivalence class of homomorphisms of $\tilde{\Omega}_x$ into G. (2) Given an equivalence

class χ of homomorphisms of $\tilde{\Omega}_s$ into G, there exist a bundle and a connection in it which realize χ. *S. Chern.*

On order

Libermann, Paulette. **Sur les variétés presque paracomplexes.** Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 5, 10 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

This lecture presents a more detailed account of material announced earlier [C. R. Acad. Sci. Paris **234**, 2517–2519 (1952); these Rev. **14**, 88], and additional facts, e.g.: the affine connection attached to an almost parahermitian structure is integrable if and only if torsion and curvature vanish. *H. Samelson* (Princeton, N. J.).

Rauch, H. E. Geodesics, symmetric spaces, and differential geometry in the large. Comment. Math. Helv. **27** (1953), 294–320 (1954).

The object of this paper is a proof of the following theorem. Let E^n be a symmetric, simply-connected Riemannian manifold of positive sectional curvature, with holonomy group H. Then there exists a constant $0<c<1$ with the following property. If a complete Riemannian manifold M^n of class C^2 with restricted holonomy group $\mathfrak{K}$ has the property that for each $P \in M^n$ there exists a linear transformation h_P of the tangent space at P onto the tangent space at a fixed point Q of E^n such that $h_P\mathfrak{K}h_P^{-1}\subset H$ and $ck(h_P\gamma)<K(P,\gamma)<k(h_P\gamma)$ for all sections γ, then the universal covering manifold of M^n is homeomorphic to E^n. Here $K(P,\gamma)$ is the curvature of M^n at P with respect to the 2-section γ, while $k(h_P\gamma)$ is the curvature of E_n at Q with respect to the 2-section $h_P(\gamma)$. The proof is accomplished by a deep and extensive use of calculus of variations methods, and in spots is difficult to understand. There is room for disagreement with the author's statements about the finality of his results. *S. B. Myers.*

Gheorghiu, Octavian Em. Sur la théorie des objets géométriques. I, II, III. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 273–284 (1952). (Romanian. Russian and French summaries)

The first part deals with the most general geometric object of two components in X_1, whose law of transformation is linear fractional of the form

$$\bar{\Omega}_1=(A\Omega_1+B\Omega_2+C)/(B\Omega_1+C\Omega_2+A)$$

and

$$\bar{\Omega}_2=(C\Omega_1+A\Omega_2+B)/(B\Omega_1+C\Omega_2+A),$$

where A, B, C are functions of $d\bar{\xi}/d\xi$. From the transitivity of the above law, the author finds that

$$A(x)=\tfrac{1}{3}(x^p+x^{a+b\theta+c\theta^2}+x^{a+b\theta^2+c\theta})$$

with similar expressions for $B(x)$ and $C(x)$, tacitly assuming the continuity of these functions, θ being a complex cube root of 1. A similar procedure is applied to a geometrical object of the third class under the conformal group in X_2. The third part is concerned with an invariant in X_{n+1}, $\bar{\Omega}(\bar{u}_i)=(A\Omega(u_i)+B)/(C\Omega(u_i)+D)$ where A, B, C, D are functions of $d\bar{u}_i/du_i$. *M. S. Knebelman.*

NUMERICAL AND GRAPHICAL METHODS

Salzer, Herbert E. Radix table for obtaining hyperbolic and inverse hyperbolic functions to many places. J. Math. Physics **32**, 197–202 (1953).

This paper does for hyperbolic functions what an earlier paper [Math. Tables and Other Aids to Computation **5**, 9–11 (1951)] does for the trigonometric functions. Methods are given for evaluating rapidly to 20 decimals either of the functions $\tanh x$ or $\tanh^{-1} t$ (not Arctanh t, which is meaningless) from a short set of tables. Other functions are then computed, if desired, by means of the well known relations connecting hyperbolic functions. The tables give $\tanh^{-1} t$ to 20 decimals for $t=\{0.1(0.1)0.9\}\times 10^{-p}$, $p=0(1)6$ and $\tanh x$ and $\frac{1}{2}\ln s$ to 20 decimals for $x=1(1)24$, $s=1.2(.2)2(1)10$. Several illustrative examples of the use of the tables are given. *J. C. P. Miller* (Cambridge, England).

Chandrasekhar, S., and Elbert, Donna. The roots of $Y_n(\lambda\eta)J_n(\lambda)-J_n(\lambda\eta)Y_n(\lambda)=0$. Proc. Cambridge Philos. Soc. **50**, 266–268 (1954).

The authors tabulate to 5D the first zero, λ_1, of the equation of the title, together with the values of

$$J_n(\lambda_1),\quad Y_n(\lambda_1),\quad J_n(\lambda_1\eta),\quad Y_n(\lambda_1\eta),\quad \frac{2}{\pi^2\lambda_1^2}\left\{\frac{J_n^2(\lambda,\eta)}{J_n^2(\lambda_1)}-1\right\}$$

for $\eta=.2$, $n=1(1)5$; $\eta=.3(.1).5$, $n=1(1)6$; $\eta=.6$, $n=1(1)8$; $\eta=.8$, $n=1(1)12$. For $\eta=.5$, $n=1(1)6$ they also tabulate the corresponding quantities for the second and third zeros. *A. Erdélyi* (Pasadena, Calif.).

Collatz, Lothar. **Graphische und numerische Verfahren.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 1–92. Verlag Chemie, Weinheim, 1953. DM 20.00.

The author presents a comprehensive account of the work on graphical and numerical methods done in Germany during the period 1939–1946. Some 222 references are listed in the bibliography (a few of these appeared earlier or later or not in Germany). The main service of the author consists in presenting a unified survey of the development in its mathematical context. The material is considered under the following headings: A. Approximate formulas; B. Numerical differentiation and integration; C. Expansion of functions (Fourier series, etc.); D. Interpolation; E. Treatment of equations; F. Ordinary differential equations; G. Partial differential equations; H. Characteristic value problems for ordinary and partial differential equations; J. Integral and functional equations; K. Bibliography. *E. Isaacson.*

Luke, Yudell L. Coefficients to facilitate interpolation and integration of linear sums of exponential functions. J. Math. Physics **31**, 267–275 (1953).

The author is concerned with interpolation formulas based on the assumption that the function $f(x)$ can be represented as a linear aggregate of exponentials in the form

$$(1)\qquad f(x)\doteq\sum_{j=0}^{n}C_j e^{-jx}$$

which clearly reduces to the classical polynomial interpolation by the transformation $y=e^{-x}$. Given, then, the ordinates f_i at the $n+1$ abscissae values x_i, $i=0, 1, \cdots, n$, the function $f(x)$ of form (1) passing through the points (x_i, f_i) is given by $f(x)=\sum_{i=0}^{n} f_i K_i^{(n)}(x)$, where

$$K_i^{(n)}(x)=\prod_{p\neq i}(e^{-x}-e^{-x_p})/\prod_{p\neq i}(e^{-x_i}-e^{-x_p}).$$

Expanding $K_i^{(n)}(x)$, we have

$$K_i^{(n)}(x)=\sum_{j=0}^{n}A_{ij}^{(n)}e^{-jx},$$

where the $A_{ij}^{(n)}$ depend on the pattern of the x_i. For the special case of equidistant $x_i=ih$, $i=0(1)n$ the $A_{ij}^{(n)}$ are tabled to 8 significant figures for $n=2(1)4$, $h=.02$, .05, .1(.1)1. Writing

$$\int_{rh}^{sh} f(x)dx=\sum_{i=0}^{n} f_i B_i^{(n)}(s, r),$$

it follows that

$$B_i^{(n)}(s, r)=\sum_{j=0}^{n}(1/j)(e^{-rjh}-e^{-sjh})A_{ij}^{(n)}.$$

Values of $B_i^{(n)}(n, 0)$ are given to 7 to 8 decimals for the above ranges of h and n. The analogue to the familiar remainder term of the classical interpolation formula is worked depending on the $(n+1)$th derivative of $f(x)$ multiplied by a factor for which upper bounds are tabled for the above n and h. *H. O. Hartley* (Ames, Iowa).

Varga, Richard S. Eigenvalues of circulant matrices. Pacific J. Math. 4, 151–160 (1954).

If the integral in the integral equation

$$U(z_j)=\lambda\int_C A(s, z_j)U(z)dq+\phi(z_j),$$

where $A(s, z_j)=d \arg (z-z_j)/dq$, is approximated by a sum for several values of the argument, the coefficient matrix of the associated system of simultaneous equations, denoted by A_{jk}, is important in the numerical calculation of the solution of the integral equation. The eigenvalues of the matrix are calculated for the case where C is an ellipse, and it is shown that the ratio of these eigenvalues to the corresponding eigenvalues of $A(s, z_j)$ approaches one-half the order of the approximating system by the use of a new method for computing the eigenvalues of a circulant matrix. A numerical example is given. *C. Saltzer* (Cleveland, Ohio).

Biedenharn, L. C., and Blatt, J. M. A variation principle for eigenfunctions. Physical Rev. (2) 93, 230–232 (1954).

This paper gives a first-order approximate formula for the eigen-vectors of a Hermitian operator on a finite-dimensional Hilbert space in terms of an orthogonal basis of given trial vectors. The result is obtained by obvious manipulations following the Rayleigh-Ritz formula for eigenvalues, with little attempt at precise statement or proof. *F. H. Brownell* (Princeton, N. J.).

Fischbach, Joseph W. Solution of least squares problems by an N step gradient method. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Tech. Note no. 719, 9 pp. (1952).

The solution of the system of linear equations $Ax+b=0$ is found by solving the related equations $A^*Ax+A^*b=0$ (where A^* is the transpose of A). The solution is obtained by the application of an N-step gradient method. The procedure is well suited for the application of high speed digital computers along the lines described by the author. The title refers to the fact that the author motivates the above solution by considering the problem of minimizing the quadratic form $J(x)=(Ax+b, Ax+b)$. *E. Isaacson.*

***Ullrich, Egon. Praxis der konformen Abbildung.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 93–118. Verlag Chemie, Weinheim, 1953. DM 20.00.

A survey of the developments in Germany of the practice and application of conformal mapping to problems of fluid dynamics and elasticity is given for the period indicated in the title. Methods of approximating conformal mapping functions are also discussed, and expositions of work done in other countries is mentioned. A bibliography is included. *C. Saltzer* (Cleveland, Ohio).

Ceschino, Francis. Critère d'utilisation du procédé de Runge-Kutta. C. R. Acad. Sci. Paris 238, 986–988 (1954).

This is a very unsophisticated discussion of the growth of the error in the application of the Runge-Kutta process to a simple type of first-order differential equation. It leads to a rather obvious result which, however, could be misleading since it fails to take account of the very important distinction between absolute and relative error. *J. C. P. Miller* (Cambridge, England).

Gautschi, Walter. Fehlerabschätzungen für die graphischen Integrationsverfahren von Grammel und Meissner-Ludwig. Verh. Naturforsch. Ges. Basel 64, 401–435 (1953).

The graphical methods of Grammel and Meissner and of Meissner-Ludwig for the integration of the differential equation $y^{(n)}=f(x, y, y', \cdots, y^{(n-1)})$ are considered in order to find bounds for the errors. For the special case $y'=f(x)$ and for some special methods applied to it, such bounds are found by Taylor's expansion. In the general case the vector $Y(x_k)=(y, y', \cdots, y^{(n-1)})$ of the exact solution for $x=x_0+kh$ (h width of step), the corresponding vector Y_k of the approximation and the error-vector E_k, the components of which are the absolute values of the components of $Y_k-Y(x_k)$ are introduced. With reference to a convex domain B of the (x, Y) space containing all the vectors $(x_k, Y(x_k))$ and (x_k, Y_k) it is assumed that $|y^{(r)}(x)|\leqq m_r$ and $|\partial f/\partial y^{(r)}|\leqq k_r$. An $n\times n$ matrix K with the rows $(0, 1, 0, \cdots, 0)$, $(0, 0, 1, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, 0, 1)$ and $(k_0, k_1, \cdots, k_{n-1})$ together with the unit matrix I and the vector $Q=(q_0, q_1, \cdots, q_{n-1})$ with $q_r=m_r+m_{r+2}+\frac{1}{3}hm_{r+1}$ are introduced so as to establish the general vector-component inequality $E_{k+1}\leqq(I+hK)E_k+\frac{1}{3}h^2Q$. This leads to convergence theorems and to another inequality of the type

$$(a_0I-b_0K)E_{r+s}\leqq\sum_{t=1}^{r}(a_tI+b_tK)E_{r+s-t}+R,$$

where the coefficients a_t, b_t and the vector R depend on the special method applied. This recurrence-type inequality is solved for the case where the equality sign holds throughout, and the solution is used for establishing an independent upper bound for the error. *H. Bückner.*

Rosen, Philip. Use of restricted variational principles for the solution of differential equations. J. Appl. Phys. 25, 336–338 (1954).

A first-order differential equation of form

$$(1) \qquad d[g(x, y)]/dx=yf(x, y, y')$$

can be regarded as the Euler equation for variation of the integral $\int_a^b[y'g(x, y)+\frac{1}{2}y^2f(x, y, y')]dx$, provided one replaces $g(x, y)$ and $f(x, y, y')$ by $g(x, u)$ and $f(x, u, u')$ during the variation and at the end sets $u=y$, and provided that $y(b)$ is given and $g(x, y)=0$ at a. This remark is used as a basis for a procedure like that of Ritz for obtaining approximate solutions for (1). The method is generalized to equations of second order and to partial differential equations; in particular, the diffusion equation $\nabla\cdot[D(c)\nabla c]=\partial c/\partial t$ and the Boltzmann transport equation are considered. *W. Kaplan* (Ann Arbor, Mich.).

Ku, Y. H. A method for solving third and higher order nonlinear differential equations. J. Franklin Inst. 256, 229–244 (1953).

The author considers the third-order ordinary differential equation

(1) $$\dddot{x}+\phi(\dot{x})\ddot{x}+f(\dot{x}, x)+f_1(x)=F(t)$$

and describes a graphical method of solving the initial-value problem. The method consists of introducing $v=\dot{x}$, $g=\ddot{x}$ and reducing (1) to the system

$$\frac{dg}{dv}=\frac{F(t)-\phi(v)g-f(v, x)-f_1(x)}{g}, \quad \frac{dv}{dx}=\frac{g}{v}, \quad \frac{dx}{dt}=v.$$

The solution is then constructed step by step by computing respective differential increments of g, v and x from the above equations in turn. No attention is paid to the effect, on the solution, of replacing differentials by divided differences or of the accumulation of the rounding-off errors. Some examples are worked and a generalisation to an nth order equation indicated. *H. O. Hartley* (Ames, Iowa).

✶Hyman, Morton Allan. On the numerical solution of partial differential equations. Thesis, Technische Hogeschool te Delft, 1953. 108 pp. (Dutch and French summaries)

This monograph is divided into four chapters and an appendix; it stresses throughout numerical methods well-suited to use with automatic computing machines. Chapter I (General considerations) discusses the convergence and stability of difference-equation solutions; it is an expansion of the author's previous discussion [O'Brien, Hyman, and Kaplan, J. Math. Physics 29, 223–251 (1951); these Rev. 12, 751]. Chapter II (Parabolic equations) discusses convergence and "extrapolation" of difference solutions. Chapter III (Elliptic equations) contains a discussion of the various known methods (direct, iterative, non-iterative, Monte Carlo) for solving boundary-value problems [Hyman, Appl. Sci. Research B. 2, 325–351 (1952); these Rev. 13, 993]. Chapter IV (Hyperbolic equations) includes a new method, designed for machine computation, for calculating gas flows with strong shock-waves; a "3-characteristics" calculation is reduced to a "2-characteristics" calculation by a device applicable to other "3-characteristics" problems. The Appendix solves a fourth-order non-linear partial differential equation occurring in turbulence theory. An extensive bibliography (1899–1953) is included.

H. Polachek (Washington, D. C.).

Juncosa, M. L., and Young, D. M. On the convergence of a solution of a difference equation to a solution of the equation of diffusion. Proc. Amer. Math. Soc. 5, 168–174 (1954).

Several sharp convergence theorems are proved [for background, see Chap. II of the paper reviewed above]. The principal theorem follows: in the region R $[0\leqq x\leqq 1, t\geqq 0]$ replace the differential problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \quad u(0+, t)=u(1-, t)=0, \quad u(x, 0+)=f(x)$$

by the difference problem

$$\frac{U_M(x, t+\Delta t)-U_M(x, t)}{\Delta t}=\frac{U_M(x+\Delta x, t)-2U_M(x, t)+U_M(x-\Delta x, t)}{\Delta x^2}$$

$$U_M(0, t)=U_M(1, t)=0; \quad U_M(j\Delta x, 0)=f(j\Delta x), \quad j=1, 2, \cdots, M-1,$$

where $0<\Delta t/\Delta x^2\leqq\frac{1}{2}$, $M\Delta x=1$. Here $f(x)$ is piece-wise continuous in $0\leqq x\leqq 1$. Then $U_M(x, t)$ can be expressed everywhere in R by an analytic formula or (for a step-by-step numerical solution) by interpolation of its values at the grid-points. In either case, for any $t_0>0$, $U_M(x, t)$ converges uniformly to $u(x, t)$ in the region $0\leqq x\leqq 1$, $t\geqq t_0>0$ as $M\rightarrow\infty$. *M. A. Hyman* (Washington, D. C.).

Juncosa, M. L., and Young, D. M. On the order of convergence of solutions of a difference equation to a solution of the diffusion equation. J. Soc. Indust. Appl. Math. 1, 111–135 (1953).

In the region R: $0<x<1$, $t>0$, the differential problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}; \quad u(0+, t)=u(1-, t)=0; \quad u(x, 0+)=f(x)$$

is replaced by the difference problem

$$\frac{v(x, t+\Delta t)-v(x, t)}{\Delta t}=\frac{v(x+\Delta x, t)-2v(x, t)+v(x-\Delta x, t)}{\Delta x^2};$$

$$v(0, t)=v(1, t)=0; \quad v(j\Delta x)=f(j\Delta x), \quad j=1, 2, \cdots, M-1$$

where $0<\Delta t/\Delta x^2\leqq\frac{1}{2}$, $M\Delta x=1$. The difference solution $v(x, t)$ can be defined at any point of R by bilinear interpolation of its values at the gridpoints. In the region R^*: $0\leqq x\leqq 1$, $t\geqq t_0>0$, with various continuity conditions on $f(x)$, the authors compute the order of max $|v(x, t)-u(x, t)|$ compared to Δx. Results throw light on efforts to improve difference solutions in practice by "extrapolation to zero grid-size" [see pp. 30–35, 85–87 of the paper reviewed second above]. *M. A. Hyman* (Washington, D. C.).

Walsh, J. L., and Young, David. On the degree of convergence of solutions of difference equations to the solution of the Dirichlet problem. J. Math. Physics 33, 80–93 (1954).

This paper derives orders of magnitude for the truncation error in the solution of the Dirichlet problem, caused by replacing Laplace's equation by the usual approximating difference equation. Restricting to square (or rectangular) regions, the authors have materially sharpened some of their previous results [J. Research Nat. Bur. Standards 51, 343–363 (1953); these Rev. 15, 562]. The effect, on the truncation error, of various restrictions on the continuity of the boundary distribution is carefully studied, in particular, the influence of jump points. *M. A. Hyman.*

Fowler, C. M. Symmetry as a factor in finite difference approximations. J. Appl. Phys. 25, 293–294 (1954).

It is shown that, in the four-point finite-difference approximation of explicit type to the heat-flow equation $\alpha T_{xx}=T_t$, with $T(0, t)=T(l, t)=0$ and $T(x, 0)=F(x)$, and with the spacings satisfying $l=N\Delta x$ and $M\alpha\Delta t=(\Delta x)^2$, the stability requirement $M\geqq 2$ can be replaced by $M\geqq 2\cos^2(\pi/2N)$ if $F(x)=F(l-x)$ and if N is even. [The author appears to overlook the fact that this situation continues to hold when the last two conditions are suppressed, and that an additional slight improvement is possible when $F(x)=F(l-x)$ if N is odd.] *F. B. Hildebrand* (Cambridge, Mass.).

Mitchell, A. R. Round-off errors in the solution of the heat conduction equation by relaxation methods. Appl. Sci. Research A. 4, 109–119 (1953).

A method is developed for assessing the magnitude of the round-off errors which arise in the relaxational solution of a stable, six-point, finite-difference approximation of implicit

type to the heat-flow equation. Explicit formulas are obtained, and their use is illustrated. *F. B. Hildebrand.*

★**Salvadori, M. G. Extrapolation formulas in linear difference operators.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 15–18. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author presents a generalisation of Richardson's h-extrapolation formula. Let $y_n(i)$ denote the approximate solution to a differential-equation problem at the argument point x_i computed at grid-width $h=c/n$ and $Y(i)$ the correct value; then it can often be assumed that

$$(1) \qquad Y(i)-y_n(i) \doteq \sum_{j=1}^{s} h^j f_j(i).$$

If we make $s+1$ determinations of the $y_n(i)$ for $s+1$ different values of $h_r=c/n_r$ $(r=1, 2, \cdots, s+1)$, then (1) provides $s+1$ approximate equations for the $s+1$ 'unknown' $c^{2j}f_j(i)$, $j=1, \cdots, s$, and $Y(i)$, and the solution resulting for $Y(i)$ is taken as an approximation to the exact $Y(i)$ and is called the extrapolated value of $Y(i)$. For a given pattern of n_r the solution $Y(i)$ of (1) is weighted averages of the computed $y_n(i)$ and the weight coefficients have been tabulated for various patterns of n_r. *H. O. Hartley.*

Batschelet, Eduard. Über die numerische Auflösung von Randwertproblemen bei elliptischen partiellen Differentialgleichungen. Z. Angew. Math. Physik 3, 165–193 (1952).

The author is concerned with numerical methods of solving the elliptic partial differential equation

$$(1) \qquad v_{xx}+v_{yy}+av_x+bv_y+cv=g$$

for an x, y region C bounded by a smooth boundary on which the solution satisfies at every point s the condition

$$(2) \qquad k(s)v_n-h(s)v=f(s)$$

and v_n denotes differentiation in the normal direction. Mixed conditions arise when for sections of the boundary $k(s)\equiv 0$ or $h(s)\equiv 0$. The solution is approximated by a finite-difference method on a square grid of mesh h at which the values v_i are made to satisfy (1) approximately by replacing $v_{xx}+v_{yy}$ by $h^{-2}(v_1+v_2+v_3+v_4-4v_0)$ where v_0 is the value at a central grid-point and v_1, v_2, v_3, v_4 those at the points adjoining it, and the first-order differentials are replaced by divided differences. The result is the familiar system of linear equations for the v_i. This is solved by a modified relaxation method which consists in alternatively reducing the residuals for the internal grid-points and those for the boundary grid-points. The convergence of this procedure is investigated and shown to get slower with decreasing h. The main theorem proved concerns an upper bound of the form $hf(x, y)$ for the difference between the approximate solution v_i at mesh h and the true solution $v(x, y)$ of (1). The improvement to order h^2 (or higher order), which is familiar with boundary conditions of the 1st and 2nd kind, is here only lightly touched upon. *H. O. Hartley.*

Kron, Gabriel. A set of principles to interconnect the solutions of physical systems. J. Appl. Phys. 24, 965–980 (1953).

This paper is concerned with the numerical solution of continuous boundary-value problems by the difference equation approximation. In particular, a problem is considered relating to Poisson's equation in two dimensions. The difference equations define a direct current electrical network. The idea is to split this network into subnetworks. By a judicious subdivision the subnetworks belong to just a few different types. The interconnection of the subnetworks solves the problem. Of course matrix inversion is required for each type of subnetwork and for the interconnection problem; however, these matrices are of appreciably lower order than the matrix of the whole network. *R. J. Duffin* (Pittsburgh, Pa.).

Böhm, Corrado. Scomposizione di un sistema di sostituzioni lineari in una successione di transfert. Univ. Roma. Ist. Alta Mat. Rend. Mat. e Appl. (5) 12, 76–89 (1953) = Consiglio Naz. Ricerche Pubbl. Ist. Appl. Calcolo no. 380 (1953).

The author is concerned with the problem of finding the least amount of storage space which must be utilized in a computing machine for storage in connection with the solution of linear algebraic equations with integer coefficients. In the course of this examination he proves a number of theorems which are useful in the solution of two major problems which he poses. Among other things he is concerned with decomposition of the matrices of his systems into products of so-called elementary ones. *H. H. Goldstine* (Princeton, N. J.).

Harmuth, Henning. Programmsteuerung einer elektronischen Rechenmaschine. Acta Physica Austriaca 7, 390–401 (1953).

The author describes a machine developed at the Institute for Low Frequency Techniques at the Technical University in Vienna. It is intended to handle classes of problems involving principally statistics. The principal organ of the machine is an electric analogue of the Galton board. Coding for the machine consists of connecting such units together in proper fashion to achieve the desired distribution functions. This code and the input-output are handled on tapes. *H. H. Goldstine* (Princeton, N. J.).

★**Rohrbach, Hans. Mathematische und maschinelle Methoden beim Chiffrieren und Dechiffrieren.** Naturforschung und Medizin in Deutschland, 1939–1946, Band 3. Angewandte Mathematik, Teil I, pp. 233–257. Verlag Chemie, Weinheim, 1953. DM 20.00.

The author was evidently occupied during World War II with the mathematical analysis of codes and ciphers for the German Foreign Office (Auswärtiges Amt) in the branch called "Sonderdienst Dahlem." In this paper he has attempted to recall the various problems and some of the results attained there and in the analogous services in the Wehrmacht and in the army. No problem is described thoroughly and no result is stated rigorously. The range of problems is impressively wide. There is a bibliography of 43 references, but of these only 5 can be found in even the best of ordinary libraries, and most of the remainder may no longer even exist.

This paper describes some of the basic methods of enciphering and some of the mathematical questions which arise in designing a cipher system. It also quickly describes some methods used by cryptanalysts, some of the machine aids used by the Germans, and some of the mathematical aids. The basic methods of enciphering described include simple and polyalphabetic substitution, and cipher machines such as the KRYHA. Also described briefly are transposition systems, additives (where the enciphering is done by adding random numbers to the plain text), code books, and more

complex cipher machines such as the Enigma and the Hagelin.

The mathematics of interest to the cryptographer includes that of permutations and their analysis into cycles. A study of a cipher machine which has irregularly moving components leads to graphs, where each possible combination of the components is plotted as a point, and two points are connected by a directed line segment if one of them represents a combination which can follow the other. Many questions of combinatorial analysis are pertinent. Generalized Fibonnacci sequences with some modulus n are sometimes used to generate "random" numbers; the question arises as to what choice of parameters will give the longest sequence. This leads into the theory of polynomials over a Galois field. Another combinatorial problem comes from the need for redundancy in order to increase the reliability of transmission. The machines described as applied to the analysis of ciphers are mostly of the punched card variety. Some special machines are described; one of them is similar to the device invented by D. H. Lehmer for the factoring of large numbers.

The mathematical theory behind these cryptanalytic methods is mainly probability and algebra. The author says that the asymptotic laws of probability play a meager role in mathematical cryptology, but more important are the Poisson limiting cases. The algebra is intimately related to combinatorial problems, as are the special methods

H. Campaigne (Washington, D. C.).

Felker, J. H. Arithmetic processes for digital computers. Electronics **26**, no. 3, 150–155 (1953) = Bell. Tel. System Tech. Publ. Monograph no. **2208**, 6 pp. (1954).

Hemphill, F. M. Suggested desk calculator operations for computing moments by the row. Biometrics **10**, 152–154 (1954).

Ter-Stepanyan, G. I. On a general property of nomograms with parallel scales for functions of several variables. Akad. Nauk Armyan. SSR. Doklady **12**, 3–8 (1950). (Russian. Armenian summary)

This paper is concerned with the problem of constructing within a given rectangle a nomogram for $\sum_{j=0}^{n} F_j(x_j)=0$, where $x_{0j} \leqq x_j \leqq x_{1j}$, using parallel scales and $n-2$ auxiliary variables. It gives (with comments) equations for the scales and eight sets of relations involving the $F_j(x_{ij})$ with the scale factors and coordinates for the location of all scales. Details of systematic application are to be given in a forthcoming work.

R. Church (Monterey, Calif.).

✶Boggio, Tommaso, e Giaccardi, Fernando. Compendio di matematica finanziaria. Operazioni di credito. G. Giappichelli, Torino, 1952. 185 pp. 1500 Lire.

After a broad introduction of various forms of interest payment and annuities certain and of methods of finding the yield of a financial investment, this textbook on compound interest gives a detailed description of amortization of loans by different schemes. An appendix gives formulae of the capital value of free shares issued by joint-stock companies.

P. Johansen (Copenhagen).

ASTRONOMY

Gjellestad, Guro. On equilibrium configurations of oblate fluid spheroids with a magnetic field. Astrophys. J. **119**, 14–33 (1954).

The Chandrasekhar-Fermi treatment of the stability of a gravitating incompressible fluid sphere with a magnetic field [same J. **118**, 116–141 (1953); these Rev. **15**, 168] is extended to oblate spheroids. Although (as shown in the earlier paper) spheres are not equilibrium configurations under the forces assumed, there is a sequence of spheroids which are.

R. G. Langebartel (Urbana, Ill.).

Chandrasekhar, S. The gravitational instability of an infinite homogeneous medium when Coriolis force is acting and a magnetic field is present. Astrophys. J. **119**, 7–9 (1954).

An infinite homogeneous density distribution is unstable according to Jeans' criterion with respect to unidirectional waves when the Coriolis force and the force resulting from a uniform magnetic field are acting simultaneously. This extends the known result that Jeans' criterion applies for either force acting alone.

R. G. Langebartel.

Sokolov, Yu. D. On some general characteristics of the behavior of a material system in the neighborhood of a singular instant of time. Dopovidi Akad. Nauk Ukrain. RSR **1951**, 227–233 (1951). (Ukrainian. Russian summary)

Consider a system of n ($\geqq 3$) particles P_i of masses m_i ($i=1, 2, \cdots, n$), which attract or repel each other, the interaction between P_i and P_j ($i \neq j$) having magnitude $m_i m_j |f(r_{ij})|$ and representing an attraction or repulsion according as f is negative or positive. It is assumed that $f(r)=dF(r)/dr$ is analytic for positive r and may have singularities at the points $r=0$ and $r=\infty$ on the real axis. The following theorems are established. Theorem 1. If the motion is regular up to but not including the instant t_1, then $\lim_{t\to t_1} \min(\underline{r}, 1/\bar{r})=0$, where $\bar{r}$ and $\underline{r}$ are the greatest and smallest of the mutual distances between the particles ac the instant t. Theorem 2. Assume that $f(r)$ is holomorphit for $-\delta<\arg r<+\delta$, its modulus is bounded for $|r|>d>0$ and $F(r)$ is bounded above in the interval $r=d>0$ and $r=+\infty$. Then $\lim_{t\to t_1} \underline{r}=0$. Theorem 3. Assume that the ratio U/I^2 is bounded above for all sufficiently large values of I^2. Then $\lim_{t\to t_1} \underline{r}=0$. Here U and I^2 denote the force function $U=\sum m_i m_j F(r_{ij})$ and the moment of inertia of the system about its center of mass $I^2=M^{-1}\sum m_i m_j r^2{}_{ij}$; M is the total mass. Theorem 4. Let x_i, y_i, z_i (coordinates of the particles with respect to the center of mass) be holomorphic functions for all values of t in the interval $t=0$ to $t=t_1$, except at t_1 itself, and let $f(r)/r$ be a holomorphic function of r^2 for $r^2=0$. Then $\lim_{t\to t_1} \bar{r}=+\infty$.

Further, on the basis of the generalized Lagrange-Jacobi equality and some supplemental assumptions, the behaviour of I^2 as $t\to t_1$ or $t\to+\infty$ is investigated [cf. Sokolov, Singular trajectories of a system of free material points, Akad. Nauk Ukrain. SSR, Kiev, 1951; these Rev. **14**, 910].

E. Leimanis (Vancouver, B. C.).

Dodd, K. N. Some calculations involving the restricted three body problem. Monthly Not. Roy. Astr. Soc. **113**, 484–492 (1953).

The author considers the equations of the restricted problem of three bodies, in space, for the case of equality of

the two finite masses, and constructs numerical solutions (with the aid of the Manchester University Electronic Computer) aiming to determine the angular momentum exchange between the bodies. He finds that, in certain particular cases, this exchange leads to a reduction in angular momentum of the binary pair consisting of the two finite masses. From these computations a numerical estimate is made of the torque experienced in a particular case of a binary star moving through a cloud of interstellar material (on the assumption that particles of this cloud behave as gas at zero absolute temperature). *Z. Kopal.*

Moser, Jürgen. Periodische Lösungen des restringierten Dreikörperproblems, die sich erst nach vielen Umläufen schliessen. Math. Ann. **126**, 325–335 (1953).

It is shown that the hypotheses of Birkhoff's theorem on the existence of periodic solutions near a given periodic solution of stable type for Hamiltonian systems with two degrees of freedom [Dynamical systems, Amer. Math. Soc. Colloq. Publ., v. 9, New York, 1927, pp. 150–159] are fulfilled in the restricted problem of three bodies, at least provided that μ, the smaller of the two finite masses, is sufficiently small. Here the given periodic solution of stable type is obtained by the Poincaré perturbation method from a suitable circular solution of the two-body problem corresponding to $\mu=0$. The Birkhoff procedure involves the determination of fixed points of the iterates of an area-preserving transformation of the form,

$$x_1=x\cos[\theta+c(x^2+y^2)]-y\sin[\theta+c(x^2+y^2)]+\cdots,$$
$$y_1=x\sin[\theta+c(x^2+y^2)]+y\cos[\theta+c(x^2+y^2)]+\cdots,$$

where the omitted terms are of degree >3 in x and y; and it is essential to know that $c\neq 0$. In the present example $\theta=\theta(\mu)$ and $c=c(\mu)$ are analytic functions of μ. The author calculates explicitly $c(0)$, which turns out to be different from zero. Hence the Birkhoff result is applicable, for μ sufficiently small, to show the existence of infinitely many periodic solutions of large period near the given periodic solution. This is the first application of Birkhoff's theorem to a non-trivial specific problem. *D. C. Lewis.*

Agostinelli, Cataldo. Sopra un caso del problema ristretto dei tre corpi più generale di quello di Hill. Boll. Un. Mat. Ital. (3) **8**, 377–384 (1953).

In the restricted problem of three bodies dealing with the sun, a planet, and a satellite of zero mass, the equations of motion of the satellite relative to the planet are written down in the usual way using rotating coordinates but without neglecting the parallax (i.e. the ratio of the planet-satellite distance to the planet-sun distance). The reciprocal of the satellite-sun distance, which appears in the potential function of these equations, obviously admits a power series development in the parallax with coefficients that are essentially the Legendre polynomials in $\cos\theta$, where θ is the angle between planet-sun line and planet-moon line. By neglecting all but the first three terms of this expansion and by neglecting also the planet's mass in comparison with the sun's mass, we obtain the Hill equations. By neglecting all but the first four terms, we obtain equations, simpler than the usual equations for the restricted problem of three bodies, but more accurate than the Hill equations, since the parallax is not completely neglected, but only powers thereof higher than the first. The paper also considers some properties of periodic solutions of these equations. The existence proofs, however, are deferred to a later paper. *D. C. Lewis* (Baltimore, Md.).

Armellini, G. Osservazioni sul problema dei due corpi di masse variabili e sopra alcune sue applicazioni alla cosmogonia. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 727–733 (1953).

According to Ambarcumyan [Trans. Internat. Astr. Union **8**, in press] and Fessenkov [ibid.] the masses of the stars decrease in the process of their development mainly as a result of corpuscular emission. In the present paper the author shows that this theory can be reconciled with the following facts, based on observation of double stars. In the case of a double star with large mass the satellite is in general quite near the principal star and its orbit has a small eccentricity; in the case of a double star with small mass the satellite is more remote from the principal star and the eccentricity of its orbit is frequently large.

The author first re-examines the validity of the two forms of the fundamental equations of the problem given by Gyldén and Levi-Cività. After concluding that under the assumption of Ambarcumyan and Fessenkov the Gyldén form of the equations is to be used, he then applies to these equations the theory of adiabatic invariants. *E. Leimanis* (Vancouver, B. C.).

Masotti, Arnaldo. Sui moti ellittici armonici. Mem. Soc. Astr. Ital. (N.S.) **25**, 45–58 (1954).

The equation of the center (difference between the true and mean anomalies) for elliptic orbits where the attractive force varies directly as the distance is studied. The points in the orbit where the absolute value of the equation of the center becomes maximum are the same points where the kinetic energy takes on its mean value with respect to the true anomaly. The circle through these points has the same area as the orbital ellipse. *R. G. Langebartel.*

Masotti, Arnaldo. Linea indicatrice della equazione del centro nei moti kepleriani. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **16(85)**, 65–81 (1952).

Let M, E, and v be the mean, the eccentric, and the true anomalies, resp., of a planet. The intersections of the radius vector with the straight lines $\theta=M$ and $\theta=E$, drawn through the center of the orbit, describe two curves which are named by the author as the indicatrix of the equation of the center and the indicatrix of the angle $(v-E)$. The author shows that the indicatrix of the equation of the center is a closed curve, symmetrical about the major axis, and differs infinitesimally from an ellipse as the eccentricity approaches zero. The sufficient condition that this curve lies inside the orbit is $e\leqq\frac{1}{2}(1+2^{3/2})^{1/2}$, which is satisfied for all planets and for many comets. Also, the indicatrix for $(v-E)$ is an ellipse. *P. Musen* (Cincinnati, Ohio).

Masotti, Arnaldo. Sui valori medi delle potenze del raggio vettore nei moti kepleriani. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **16(85)**, 273–277 (1952).

The three mean values α, β, γ of a function $G(r)$ of the radius vector of a planet are defined as integrals with respect to the mean, the eccentric, and the true anomalies, taken over the interval $(0, 2\pi)$. The author establishes the inequalities connecting α, β, and γ for the case in which $G(r)$ is a monotonic function of r. A special case $G=r^n$ is investigated. A graph is given, showing the mutual relation of the three mean values. *P. Musen* (Cincinnati, Ohio).

Bucerius, H. Bahnbestimmung als Randwertproblem. IV. Astr. Nachr. **281**, 97–106 (1953).

Continuing a program described in three previous papers [Astr. Nachr. **278**, 193–203, 204–216 (1950); **280**, 73–82

(1951); these Rev. 12, 753; 14, 211], the author investigates further the convergence of several methods of orbit determination; regions of convergence and divergence are found and tested on observed orbits. *W. Kaplan.*

Kikuchi, Sadaemon. The distribution function of velocities of the stellar system in steady state. Sci. Rep. Tôhoku Univ., Ser. 1. 36, 63–72 (1952).

It is know that if f denotes the distribution function of velocities in a stellar system in a steady state, then from Liouville's equation it follows that f can be an arbitrary function of the five integrals $I_1, \cdots, I_5$ of the equations of motion (Jeans's theorem). On the other hand, the density of stars is given by

$$\rho(x_i) = \iiint f(x_i, p_i)\,dp_1 dp_2 dp_3,$$

where the integration is effected over the accessible velocity (p_i) space. The author shows by means of examples that if f is a function of only three integrals I_1, I_2, I_3 (say), then the foregoing equation written in the form

$$\rho(x_i) = \iiint f(x_i, p_i)\frac{\partial(p_1, p_2, p_3)}{\partial(I_1, I_2, I_3)}\,dI_1 dI_2 dI_3,$$

provides an integral equation which under certain circumstances can give useful information. *S. Chandrasekhar.*

Kikuchi, Sadaemon. Stellar dynamical meanings of Schürer's transformation. Sci. Rep. Tôhoku Univ., Ser. 1. 37, 302–306 (1953).

Schürer [Astr. Nachr. 273, 230–242 (1943); these Rev. 6, 244] showed how Chandrasekhar's results [Principles of stellar dynamics, Univ. of Chicago Press, 1942; these Rev. 4, 57] in the dynamics of nonsteady stellar systems can be derived from those valid for systems in a steady state by a certain transformation. In this paper the author shows that the transformation in question is a contact transformation. Thus, let ξ_i and v_i $(i=1, 2, 3)$ be a set of new rectangular co-ordinates and velocities obtained from the Cartesian co-ordinates x_i and the velocities u_i by the contact transformation

$$d\Psi = \sum_i (v_i d\xi_i - u_i dx_i),$$

where

$$\Psi = \sum\left(\varphi x_i v_i - \frac{1}{2\varphi}\frac{d\varphi}{dt}x_i^2\right)$$

and φ is an arbitrary function of time. The relation between the two corresponding Hamiltonians H and K is given by

$$(*)\quad K = H + \frac{\partial\Psi}{\partial t} = \tfrac{1}{2}\sum_i u_i^2 + V(x_i, t) + \sum_i\left\{\frac{d\varphi}{dt}x_i v_i - \frac{d}{dt}\left(\frac{1}{2\varphi}\frac{d\varphi}{dt}\right)x_i^2\right\}.$$

Also, since

$$u_i = \frac{\partial\Psi}{\partial x_i} = \varphi v_i - \frac{1}{\varphi}\frac{d\varphi}{dt}x_i \quad\text{and}\quad \xi_i = \frac{\partial\Psi}{\partial v_i} = \varphi x_i,$$

we can rewrite (*) in the form

$$K = \varphi^2\left\{\tfrac{1}{2}\sum_i v_i^2 - \frac{1}{2\varphi}\frac{d^2\varphi}{dt^2}\sum_i \xi_i^2 + \frac{1}{\varphi^2}V\left(\frac{\xi_i}{\varphi}, \tau\right)\right\}$$

where $dt = \varphi^2 d\tau$. The foregoing equations are equivalent to those underlying Schürer's transformation.

S. Chandrasekhar (Williams Bay, Wis.).

Suda, Kazuo. On the rotation problem of completely degenerate stellar configurations. Sci. Rep. Tôhoku Univ., Ser. 1. 37, 307–319 (1953).

The slow uniform rotation of completely degenerate stellar configurations is studied by standard perturbation methods. The functions (two for each assigned value of the central density) describing the perturbed configuration are numerically integrated; and the resulting ellipticities are tabulated and illustrated. *S. Chandrasekhar.*

Prasad, Chandrika. Radial oscillations of a composite model. Proc. Nat. Inst. Sci. India 19, 739–745 (1953).

The author considers first-order adiabatic pulsations of the "generalized Roche model", consisting of homogeneous but compressible fluid surrounded by an envelope of uniform small density. He finds that the existence of small radial oscillations does not place any restrictions on the ratios of the densities or fractional dimensions of the core and its envelope, or on the ratio of specific heats of their materials. The periods and relative amplitudes of three lowest modes of oscillation have been determined for five different ratios of the relative dimensions and densities of the core and the envelope; and it is shown that the amplitudes of oscillation vary throughout the interior.

Z. Kopal (Manchester).

Huang, Su-Shu. Pulsations of a rotating star. Ann. Astrophysique 16, 315–320 (1953).

The author applies the Ritz method to the equation of motion governing the radial oscillations of a rotating star to derive a formula for the oscillation frequency. His result is the same as that obtained by Ledoux [Astrophys. J. 102, 143–153 (1945); these Rev. 7, 225] by an application of the virial theorem. The author also points out the relation of his method to the somewhat different variational method employed by Cowling and Newing [ibid. 109, 149–158 (1949); these Rev. 10, 746] for the same problem.

R. G. Langebartel (Urbana, Ill.).

Chandrasekhar, S., and Limber, D. Nelson. On the pulsation of a star in which there is a prevalent magnetic field. Astrophys. J. 119, 10–13 (1954).

Employing the form of the virial theorem derived in an earlier paper [Chandrasekhar and Fermi, same J. 118, 116–141 (1953); these Rev. 15, 168] the authors obtain an approximate formula for the frequency of radial pulsation of a gaseous star with a magnetic field.

R. G. Langebartel (Urbana, Ill.).

Horak, Henry G., and Lundquist, Charles A. The transfer of radiation by an emitting atmosphere. II. Astrophys. J. 119, 42–50 (1954).

In continuation of the first author's previous investigation [same J. 116, 477–490 (1952); these Rev. 14, 804], the transfer of radiation by plane-parallel atmosphere containing a linear distribution of emission sources is considered for the isotropic case. The corresponding integro-differential equations of transfer are set up, together with boundary conditions corresponding to both finite and semi-infinite atmospheres, and exact solutions are presented for the case of isotropic scattering with the albedo $\omega_0 < 1$ as well as for the "conservative" case when $\omega_0 = 1$. A method is developed by which the exact field of radiation at any optical depth can be obtained. *Z. Kopal* (Manchester).

Wrubel, Marshal H. On the decay of a primeval stellar magnetic field. Astrophys. J. 116, 291–298 (1952).

The author sets up the Maxwell equation

$$\nabla^2 E = 4\pi\sigma\frac{\partial E}{\partial t} \tag{1}$$

for the stellar electric field strength E, where from Cowling [Monthly Not. Roy. Astr. Soc. 105, 166–174 (1945)] he writes for the electrical conductivity $\sigma = 5.87\times10^{-15}T_c^{3/2}\theta^{3/2}$ with T_c=central temperature and $T_c\theta$=temperature at distance r from center. Since only axial symmetrical solutions for E are sought, (1) reduces to an equation for the azimuthal component E', which in polars r, $\mu=\cos\theta$ and ϕ can be written as

$$E'_{rr}+\frac{2}{r}E_r'+\frac{(1-\mu^2)}{r^2}E'_{\mu\mu}-\frac{2\mu}{r^2}E_\mu'-\frac{E'}{r^2(1-\mu^2)}=4\pi\sigma E_t'. \tag{2}$$

The solution of (2) is sought in the form $E'=g(r)P_n(\mu)e^{-t/\tau}$, where $P_n(\mu)$ is the nth degree Legendre polynomial and τ the required decay time to fraction $1/e$. If R is the radius of the star, $x=r/R$ and $p(x)=xg(xR)$, then (2) reduces to

$$p_{xx}+\left[\Gamma\theta^{3/2}(x)-\frac{n(n+1)}{x^2}\right]p=0, \tag{3}$$

where the eigenvalues Γ_i=(physical constants)$/\tau_i$ provide the eigenvalues for the τ_i, and $\theta(x)$ has been tabulated by M. Schwarzschild [same J. 104, 203–207 (1946)]. The boundary condition, at $x=1$, for the solution $p(x)$ of (3) can be derived from physical considerations as

$$p_x(1)+np(1)=0. \tag{4}$$

For trial values of Γ, (3) was expanded as a power series in x, the left hand side of (4) for $x=1$ and roots Γ of (4) computed by inverse interpolation of $p_x(1)+np(1)$ to 0.

H. O. Hartley (Ames, Iowa).

Whitham, G. B. The propagation of weak spherical shocks in stars. Comm. Pure. Appl. Math. 6, 397–414 (1953).

The author considers the propagation of spherical waves of finite amplitude through the interiors of self-gravitating gas spheres in polytropic equilibrium, with the aim of investigating the conditions under which a relatively weak disturbance near the center can gain strength by outward propagation. He departs from linearized equations of the problem, and constructs their solution in the form of an expansion which is valid provided that the velocity of sound $A(r)$ in the equilibrium configuration is such that $A'(r)=f(A/r)$, where r denotes the distance from the center. This solution is then improved by a method used elsewhere [cf. Whitham, same Comm. 5, 301–348 (1952); these Rev. 14, 330] to give a corrected (non-linear) theory of the propagation of weak disturbances. The region of validity of so corrected a solution includes the head shock, whose propagation may thus be studied. Near the surface of a polytropic gas sphere of finite dimensions the above conditions restricting the variation of $A(r)$ is bound to break down; but an alternative approximate solution of the linearized equations is obtained, and the behavior of the shock studied.

It is shown that the pressure jump across the shock tends to zero as $A\to0$; but that the shock strength and the velocity of mass motion immediately behind the shock may increase or decrease in the course of outward propagation, in accordance with the criteria which are explicitly formulated in terms of the equilibrium properties of the respective configuration.

Z. Kopal (Manchester).

RELATIVITY

Teisseyre, Roman. Note on the problem of coordinate conditions and equations of motion in general relativity theory. Acta Phys. Polonica 13, 45–49 (1954). (Russian summary)

The author shows by calculating the equations of motion and the field equations to the fifth order of approximation without recourse to coordinate conditions that in the "new approximation" method the equations of motions in the Newtonian approximation and in the next approximation are uniquely determined by adopting the Newtonian potential as the solution of the first field equation.

A. H. Taub (Urbana, Ill.).

Ikeda, Mineo. On a five dimensional representation of the electromagnetic and electron field equations in a curved space-time. Progress Theoret. Physics 10, 483–498 (1953).

The author studies the geometry of 4-dimensional Riemannian spaces which can be immersed in a 5-dimensional flat space. He discusses Maxwell's and Dirac's equation in such spaces, particularly in the cosmological model of de Sitter and Einstein.

A. E. Schild (Pittsburgh, Pa.).

Schrödinger, E. Electric charge and current engendered by combined Maxwell-Einstein-fields. Proc. Roy. Irish Acad. Sect. A. 56, 13–21 (1954).

The author uses a set of successive approximations of the field variables g_{ij} and Γ^i_{kl} entering into the generalised theory of gravitation to show that in the second order the equations describing the vanishing of the four-current vector are not compatible with the generalized field equations

$$g_{\underset{+-}{ik};l}=0,\quad \Gamma^l_{\underset{\vee}{kl}}=0,\quad R_{kl}+\tfrac{2}{3}(\Gamma_{l,k}-\Gamma_{k,l})=0.$$

Thus these field equations state that in general the simultaneous presence of an electromagnetic and gravitational field in a region of space-time entails a current field within that region.

A. H. Taub (Urbana, Ill.).

Rao, B. R., and Gupta, D. P. The inertial field of a charged particle. Proc. Nat. Inst. Sci. India 19, 729–738 (1953).

A spherically symmetric solution of Einstein's 1950 unified field theory is worked out and the gravitational metric is found to be

$$ds^2=\left(1+\frac{e^2}{r^4}\right)dt^2-dr^2-r^2(d\theta^2+\sin^2\theta\,d\varphi^2).$$

It corresponds to a static and spherically symmetric electromagnetic field due to a point-charge e. In the course of the solution it is necessary to choose between equating to zero the rt-component or the $\theta\varphi$-component of the fundamental tensor, both components being antisymmetric. The second choice leads to no solution; the first one does and the rt-component is then identifiable with the electric intensity. The energy of the electromagnetic field round the charge is calculated by using a definition of the energy-tensor of the

same form as in Einstein's 1916 theory. The energy-distribution is however found to be weaker than in the latter theory.
G. C. McVittie (Urbana, Ill.).

Pham, Mau Quan. Mouvements permanents d'un fluide parfait thermodynamique. C. R. Acad. Sci. Paris **238**, 324–325 (1954).

Using the same general approach as in previous papers [same C. R. **236**, 2299–2301; **237**, 22–24 (1953); these Rev. **14**, 1134, 1135], the author investigates permanent motions of a perfect thermodynamic fluid in general relativity, a permanent motion being defined as one in which space-time is stationary and its group of isometries leaves invariant the velocity, pressure and temperature of the fluid.
J. L. Synge (Dublin).

Costa de Beauregard, Olivier. Dynamique relativiste des n points et statique classique des n fils. C. R. Acad. Sci. Paris **237**, 1395–1397 (1953).

The author first points out the formal analogy between the dynamical theory of a moving charged particle with spin and the theory of static equilibrium of stiff wires. He proposes to modify and interpret the Wheeler-Feynman dynamical theory of n interacting charged particles in terms of ideas suggested by this formal analogy. *A. H. Taub.*

Shibata, Takashi. Fundamental group of transformations in special relativity and quantum mechanics. J. Sci. Hiroshima Univ. Ser. A. **16**, 61–66 (1952).

Shibata, Takashi. Some properties of Lorentz transformations. J. Sci. Hiroshima Univ. Ser. A. **16**, 285–290 (1952).

Shibata, Takashi. Definition of momentum and mass as an invariant vector of the new fundamental group of transformations in special relativity and quantum mechanics. J. Sci. Hiroshima Univ. Ser. A. **16**, 487–496 (1953).

Shibata, Takashi. Some results deduced from the new fundamental group of transformations in special relativity and quantum mechanics. J. Sci. Hiroshima Univ. Ser. A. **17**, 67–73 (1953).

The author of this series proposes to replace the full Lorentz group of space-time transformations by a special 3-parameter sub-group G_3, and to develop physical laws which are invariant under this more restricted group. [Actually, the sub-group considered is that which leaves a given null-ray $(1, cd^i)$ of Minkowski space-time invariant, where the d^i are the direction cosines of the spatial projection of the invariant ray.] The formal apparatus developed in the first two papers is applied in the third to a determination of the possible forms of dependence of the energy-momentum vector of a particle on its velocity. In addition to the usual special relativistic term there appear two further terms, one of which is eliminated in the fourth paper by requiring axial symmetry about the given direction d^i. [The remaining new term is in fact interpretable as a beam of radiant energy of amount nc^2 along the given null-ray.] In the last paper the author shows that a light beam in the direction d^i suffers no aberration, in the sense that its coordinate velocities are the same for all reference frames in the group. [The restriction to observers moving transverse to the direction of the beam is in fact unnecessary, as the null-ray is invariant under the full G_3.]
H. P. Robertson (Pasadena, Calif.).

Hély, J. La dynamique du point matériel doué de spin. Mem. Artillerie Française **26**, 859–871 (1952).

The author sets up, within the framework of special relativity, equations describing the motion of a particle endowed with intrinsic angular momentum, as given by the spatial components of an antisymmetric tensor C^{ij}, and acted upon by an internal torque M^{ij} in addition to the usual ponderomotive force. On imposing certain conditions on these quantities, he obtains an equation for the total energy containing contributions due to the intrinsic elements. In the second section the transition to wave mechanics is effected by replacing the dynamical variables by Dirac operators, leading to terms including the known field-spin interaction terms, and on factorization to the Dirac equations. The remainder of the paper is devoted to a discussion of the vector and pseudo-vector operators associated with spin, and to remarks on the classical interpretation of the formalism.
H. P. Robertson (Pasadena, Calif.).

MECHANICS

★Destouches, Jean-Louis. Principes de la mécanique classique. Centre d'Etudes Mathématiques en Vue des Applications. C. Physique mathématique, vol. I. Centre National de la Recherche Scientifique, Paris, 1948. 137 pp.

This book has three chapters: I. The principles of classical mechanics; II. Fundamental theorems of classical mechanics; III. First integrals. It ends with some simple examples, a bibliography and an index. This is the first of a series devoted to various aspects of mechanics—Newtonian mechanics, statistical mechanics, and wave mechanics (non-relativistic and relativistic)—and the aim of the present volume is to fix very precisely the principles of classical mechanics to serve as a point of departure for the study of more modern theories. As the author remarks, it is extremely difficult to arrive at a perfectly satisfactory enunciation of these principles; the plan adopted in this book is the result of much discussion by the commission of mechanics of the Centre d'Etudes Mathématiques en Vue des Applications. Newtonian mechanics, as here presented, deals with systems of particles interacting with equality of action and reaction (continua are not considered), and the principles are basically the same as one is accustomed to meet in mathematical presentations of mechanics, but here set out very carefully and methodically. Force is treated as a derived concept, not a primitive one. The book contains some references to physics, but these are not integrated with the mathematical argument, and, in view of the expressed aim of the book, the reviewer was a little disappointed not to find any light thrown on that dark and confusing region where physics and mathematics meet. Indeed the confusion is increased by the remark of the author that some writers distinguish real classical kinematics, based on the principles enunciated in this book, from abstract classical kinematics in which one makes no appeal to physical frames of reference nor to the notion of physical time and in which one confines oneself to the consideration of movements (in fact, sets of positions, continuous and twice differentiable functions of a numerical variable t without precise physical meaning) with respect to a geometrical triad of axes in a purely abstract study. This

remark seems to imply that Newtonian mechanics is physical reality in the view of the author; if so, what of relativistic mechanics or quantum mechanics? The reviewer's opinion is that no mathematical model should be confounded with physical reality, and that the scheme so carefully expounded in this book is essentially a mathematical construct, suggested by physical reality but not physical reality by any means. *J. L. Synge* (Dublin).

Storchi, Edoardo. Sul principio dell'azione potenziale stazionaria. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 771–778 (1953).

For a dynamical system possessing potential energy V and kinetic energy T homogeneous quadratic in the velocities the author gives a very simple proof that Hamilton's principle $\delta\int(T-V)dt=0$ implies the principle of stationary action $\delta_a\int 2Tdt=0$, $\delta_a E=0$; here δ and δ_a mean, respectively, synchronous and asynchronous variations and $E=T+V$. (The author refers to the principle of stationary action as Hölder's principle.) This is followed by a discussion of variational principles of the form $\delta_a\int f(T, V)dt=0$, the varied motion being restricted by a condition $\delta_a Q=0$ where Q is a function of T, V and dt, and the following theorem is established: Among all varied motions which keep the extreme configurations fixed and satisfy $\delta_a(E^{\lambda-1}dt^\lambda)=0$ in the variation from the natural motion (λ is any constant), the natural motion is that which gives a stationary value to the integral

$$A=\int(2T-\lambda E)dt=\int[(2-\lambda)T-\lambda V]dt.$$

For $\lambda=0$ this reduces to Hölder's principle and for $\lambda=1$ to Hamilton's principle. For $\lambda=2$ it gives $\delta_a\int Vdt=0$ under the condition $\delta_a(Edt^2)=0$; this is the principle of stationary potential action to which the title of the paper refers. *J. L. Synge* (Dublin).

Longhini, Pedro. The state of velocity and the circumference of inflexion in plane motion. An. Soc. Ci. Argentina **156**, 25–33 (1953). (Spanish)

The author studies some properties of rigid motions in a plane, particularly ones concerning the circumference of inflection Γ. This is the locus of points in the moving plane which, at the instant considered, are at points of inflection of their trajectories. It is shown that Γ is determined by the instantaneous velocity field, and by the radii of curvature (at C) of the paths of the center of rotation C in the fixed and moving planes. *L. A. MacColl.*

Vâlcovici, V. Sur les conditions initiales en mécanique. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 303–310 (1951). (Romanian. Russian and French summaries)

The author observes that in classical mechanics one asserts that a material point can be assigned arbitrary initial state (position and velocity) and that the motion is then uniquely determined. He remarks that there are infinitely many solutions to the problem of finding a force required to bring a given particle from one state at one time to another at another. Second, he observes that for a force field which is insufficiently smooth, there may fail to exist a solution, or there may exist more than one. He is willing to discard the former possiblity as without interest, but he points out that there are interesting and possibly realizable examples of the latter. He shows that a mathematical extension of Newton's first law can be used in this case to select a single one of the possible solutions. Thus at points where more than one solution of the initial value problem exists, the first law of Newton is not a consequence of the second. *C. Truesdell* (Bloomington, Ind.).

Moser, Jürgen. Über periodische Lösungen kanonischer Differentialgleichungssysteme. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys. Chem. Abt. **1953**, 23–48 (1953).

A study is made of the solutions of a Hamiltonian system with two degrees of freedom in the neighborhood of a periodic solution C. The Hamiltonian H is assumed to be time-independent and to satisfy differentiability conditions (not analyticity). It is shown that, if an integral G, independent of H, exists which is 0 on C and otherwise positive, then, for fixed H, the system can be put in the form (1) $du/d\theta=-\lambda v$, $dv/d\theta=\lambda u$, where λ depends on $G=u^2+v^2$ and θ is an angular variable. Thus for fixed G the equations describe a flow on a torus whose character depends on the rationality of λ. A similar result was obtained by G. D. Birkhoff [Acta Math. **43**, 1–119 (1922)] by formal power series, but the convergence of the series was not settled. The author also studies the case when H depends on a parameter μ and the conditions described above hold for $\mu=0$. It is shown that, under appropriate conditions, periodic solutions exist near C for sufficiently small μ. This is applied to the restricted problem of 3 bodies. It is shown by an example that the reduction to the normal form (1) is not necessarily possible if the system is not canonical. *W. Kaplan* (Ann Arbor, Mich.).

Zeuli, Tino. Sistemi dinamici corrispondenti con forze funzioni lineari delle velocità. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **86**, 59–81 (1952).

Correspondence of dynamical systems in the sense of Painlevé is considered in which the forces are linear (in general, non-homogeneous) functions of the velocities with coefficients depending only on the coordinates. The author is able to reduce the study of the correspondence of such systems to the special case treated by Levi-Cività, in which the forces vanish. Necessary and sufficient conditions are obtained that a pair of such systems should correspond; whence the author determines those systems which admit certain types of first integrals quadratic in the velocities. *D. C. Lewis* (Baltimore, Md.).

Colombo, G. Un teorema di dinamica ed una sua applicazione al moto di un corpuscolo elettrizzato in presenza di un dipolo. Rend. Sem. Mat. Univ. Padova **22**, 207–222 (1953).

A Hamiltonian system of two degrees of freedom, for which

$$H=\tfrac{1}{2}(p_1^2+p_2^2)-U(q_1, q_2),$$

where $U_{q_1}(q_1^0, q_2)=0$ and $U_{q_1q_1}(q_1^0, q_2)<0$ for some constant q_1^0, is supposed to satisfy also the condition that, for some constant h, the manifold whose equation is $2[h+U(q_1, q_2)]-p_2^2=0$ in Euclidean (q_1, q_2, p_2) three-dimensional space contains a convex closed surface in whose interior $2(h+U)-p_2^2>0$. It is assumed that q_1^0 represents an interior point of the projection of this surface on the q_1-axis, so that there are periodic solutions corresponding to $q_1=q_1^0$ with $H=h$. Finally we assume that the characteristic exponents of these periodic motions are of a certain irrational type. Then it is proved that there exists at least

one other periodic solution with $H=h$ in which $q_1(t)$ has just one relative maximum and one relative minimum in a period. The method of proof is to study the fixed points of a certain continuous transformation of the interior and boundary of a topological circle into itself. On account of the irrationality of the characteristic exponents, it is shown that the transformation has no fixed points on the boundary. Hence, it must have at least one fixed point in the interior, which is then easily seen to lead to a periodic solution of the required type. An example of this phenomenon is given as indicated by the title. *D. C. Lewis* (Baltimore, Md.).

Colombo, G. Sul moto di due corpi rigidi pesanti collegati in un punto, di cui uno ha un punto fisso. Rend. Sem. Mat. Univ. Padova 22, 305–312 (1953).

A system S consisting of two rigid bodies S_1 and S_2 is considered under the influence of gravity. S_1 is supposed to have an absolutely fixed point, while S_2 has a point fixed relative to S_1. The author studies necessary and sufficient conditions that both bodies can rotate with uniform angular velocities about fixed axes. In most cases, of course, S must rotate as a single rigid system. Properties of these motions are developed with the help of recent but well known geometrical methods due to Signorini. *D. C. Lewis.*

Capriz, Gianfranco. Precessioni di un giroscopio pesante con armatura asimmetrica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 12, 229–236 (1953).

The system considered has four degrees of freedom and consists of: (1) a rigid body S' (called an "armature") of mass m', with center of gravity at G', and free to rotate about a fixed point O different from G'; (2) a second rigid body S (called a "gyroscope") of mass m, with center of gravity G and free to turn about an axis of symmetry of S, fixed in S' and passing through O but not through G'. The author determines all motions of the system which correspond to a uniform rotation of S' about some fixed axis through O, thus completing the results already obtained by Signorini when O, G, and G' are collinear. *D. C. Lewis.*

Alfieri, Leandro. Risoluzione di un problema comprendente quello di Staude. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 12 (1953), 367–372 (1954).

The problem of the title is to determine all possible uniform rotations of a solid about a fixed point O different from the center of gravity G, when the acting system of forces is equivalent to a constant force applied to a point P different, in general, from both G and O. *D. C. Lewis.*

Četaev, N. G. On stability of rotation of a rigid body with a fixed point in Lagrange's case. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 123–124 (1954). (Russian).

The necessary and sufficient condition for stability of a sleeping top is standard textbook material ($C^2r_0^2 \geqq 4Amgs$). The author succeeds in proving its sufficiency by a less direct method employing Liapounoff's criterion (existence of a positive definite function of the perturbations). Routh [Advanced rigid dynamics, 6th ed., Macmillan, London, 1905, arts. 214 and 214c] solves the problem for uniform precession (not just rotation) about the vertical, the moments of inertia and the position of the mass center being unrestricted. *A. W. Wundheiler* (Chicago, Ill.).

Hydrodynamics, Aerodynamics, Acoustics

Vâlcovici, V. Sur le mouvement tourbillionnaire des fluides barotropes. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 541–545 (1952). (Romanian. Russian and French summaries)

The author begins by attributing to certain Russian writers some classical formulae of Euler, Lagrange, and Lamb concerning steady barotropic flow of inviscid fluids subject to conservative extraneous force. His result may be put as follows: the superposition of a uni-directional steady isochoric motion on an irrotational isochoric motion is circulation-preserving if and only if the former is uniform or the latter is zero or the latter is plane and the former is linear in one co-ordinate in that plane. *C. Truesdell.*

Kumar De, Kamini. On an extension of Blasius theorem. Bull. Calcutta Math. Soc. 45, 121–124 (1953).

In steady two-dimensional motion of an inviscid incompressible fluid it is proved that the force and moment on a fixed cylinder are given by

$$X-iY=\tfrac{1}{2}i\rho\int_C (u-iv)^2dz,\quad M=\mathrm{Re}\left[-\tfrac{1}{2}\rho\int_C (u-iv)^2 z\,dz\right].$$

These results hold whether the motion is irrotational or rotational. In the case of rotational motion $u-iv$ is not an analytic function of z, but since $\bar{z}$ can be expressed in terms of z on the boundary C of the cross-section, this difficulty disappears. *L. M. Milne-Thomson* (Greenwich).

***Roseau, Maurice. Contribution à la théorie des ondes liquides de gravité en profondeur variable.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 275, ii+91 pp. (1952). 1300 francs.

The paper contains the detailed developments and proofs of work announced in C. R. Acad. Sci. Paris **231**, 1212–1214 (1950); **232**, 211–213, 303–306, 479–481; **233**, 844–845, 916–917 (1951); **234**, 297–299 (1952); and already reviewed in these Rev. **12**, 870, 869; **14**, 215. *J. J. Stoker.*

Fortier, André, et Kravtchenko, Julien. Sur quelques propriétés des solutions des équations de Navier. Cahiers de Physique no. **42**, 64–74 (1953).

A survey, with proofs, of several known results based on the dissipation function in the theory of viscous fluids; e.g. Bobyleff's theorem, and the uniqueness of slow (Stokes) flows. *D. Gilbarg* (Stanford, Calif.).

Kravtchenko, Julien, de Saint-Marc, Gaston Sauvage, et Boreli, Mladen. Sur les singularités des écoulements plans permanents des liquides en milieux poreux. C. R. Acad. Sci. Paris **238**, 209–211 (1954).

Flow of heavy liquid in a domain bounded by an open polygon γ closed by a curve C is considered. Special attention is given to singularities caused by the vertices of γ. More specifically, the authors are concerned with simplifying the reasoning of Polubarinova-Kochina [Theory of movement of ground water, Gostehizdat, Moscow, 1952, Chap. 7; these Rev. **15**, 71] with respect to these singularities. *R. E. Gaskell* (Seattle, Wash.).

Bland, D. R. Mathematical theory of the flow of a gas in a porous solid and of the associated temperature distributions. Proc. Roy. Soc. London. Ser. A. **221**, 1–28 (1954).

This paper is concerned with the flow of a gas in a porous solid and the heat transfer between them. The analysis is

based upon Darcy's law, the perfect gas law, and a set of three heat-transfer equations which are derived by the author. Steady-state cases involving one independent space variable are specialized from these general equations, and their general solutions are obtained for linear flow as well as for radial flow in cylinders and spheres. Proceeding further, the author finds the steady state axially symmetric flow and temperature in a laterally insulated porous cylinder by using the relaxation method. Finally, he considers the unsteady linear flow in a regenerator which alternately carries hot and cold gases, and writes the solution in Fourier series form. *R. E. Gaskell* (Seattle, Wash.).

Lakshmana Rao, S. K. On a class of viscous compressible flows. J. Indian Inst. Sci. **36**, 33–35 (1954).

A motion is said to admit a flexion-potential H if its vorticity ζ is such that $\nabla\times\zeta=\nabla H$. A theorem, due ultimately to Craig [Amer. J. Math. **3**, 269–293 (1880), pp. 272–273], asserts that a flow of a viscous fluid of uniform density and viscosity, subject to conservative extraneous force, is circulation-preserving if and only if it admits a flexion-potential. In a rather circuitous way the author observes that this theorem can be extended to barotropic flow of a compressible fluid, the acceleration-potential then being given by

$$\Omega+\int dp/\rho+\tfrac{1}{2}q^2-(\lambda+\mu)\vartheta/\rho+\mu H/\rho,$$

where the notations are classical. The rest of the author's results are immediate from well-known properties of circulation-preserving motions [cf., e.g., the reviewer's Kinematics of vorticity, Indiana Univ. Press, 1954, §49², §75, Ch. IX]. *C. Truesdell* (Bloomington, Ind.).

Nigam, Swami Dayal. Note on the boundary layer on a rotating sphere. Z. Angew. Math. Physik **5**, 151–155 (1954).

The author has extended Howarth's treatment of this problem [Philos. Mag. (7) **42**, 1308–1315 (1951); these Rev. **13**, 506] by employing a series solution somewhat different from the one discussed, but not worked out, by Howarth. The ordinary differential equation satisfied by the first nine functions involved in the series expansions of the three velocity components are set up. These are solved approximately by expressing the functions as fourth-degree polynomials and integrating across the boundary layer. The results show radial inflow from the pole ($\theta=0°$) to $\theta=54°\,45'$ and outflow between this latitude and the equator ($\theta=90°$). Nothing can be said about the convergence of the series solution near the equator. Thus, the results confirm Howarth's conjectures regarding the radial flow better than his own approximate solution. *W. R. Sears* (Ithaca, N. Y.).

Fadnis, Bhaskar Sadashiv. Boundary layer on rotating spheroids. Z. Angew. Math. Physik **5**, 156–163 (1954).

The methods of the paper reviewed directly above are applied to rotating prolate and oblate spheroids. *W. R. Sears* (Ithaca, N. Y.).

Truckenbrodt, E. Ein Quadraturverfahren zur Berechnung der Reibungsschicht an axial angeströmten rotierenden Drehkörpern. Ing.-Arch. **22**, 21–35 (1954).

This is an attempt to extend the work of Schlichting for the laminar boundary layer [Ing.-Arch. **21**, 227–244 (1953); these Rev. **15**, 477] to the turbulent case. Momentum-integral methods are used throughout. In the absence of a theory of the turbulent layer, empirical results are adopted. These are based principally on an experimental result of Young and Booth [Aeronaut. Quart. **3**, 211–229 (1951)], namely, that in cylindrical flow the cross-flow boundary layer is independent of the axial flow. (This result is erroneously attributed also to Rott and Crabtree [J. Aeronaut. Sci. **19**, 553–565 (1952)]; actually these authors only discussed critically the Young-Booth result.) It is also assumed that there is a similarity between the velocity profiles of the azimuthal and meridional flow components, so that the resultant wall shear is in the direction of the resultant relative potential flow. On the basis of these assumptions and a power law relating wall shearing stress to momentum thickness, the growth and separation of boundary layers, the shear torque, etc., are calculated. Detailed results are given for a rotating disk, sphere, spheroid, and half-body. Comparisons with the laminar cases and (for the sphere) with experiment are included. [The reviewer remarks that recent experiments at Cornell University have failed to confirm the Young-Booth result.] *W. R. Sears* (Ithaca, N. Y.).

Chu, Sheng To, and Tifford, A. N. The compressible laminar boundary layer on a rotating body of revolution. J. Aeronaut. Sci. **21**, 345–346 (1954).

Iacob, C. Etude comparée des variantes de la méthode approchée de S. A. Tchaplyguine dans le problème de l'écoulement subsonique autour du cylindre circulaire. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 293–302 (1951). (Romanian. Russian and French summaries)

This is similar to the author's earlier paper [C. R. Acad. Sci. Paris **223**, 714–716 (1946); these Rev. **8**, 236] summarizing his work on approximate solutions of plane compressible flow in the hodograph plane and comparing it with the work of other authors. The three variants considered are based on the following approximations to the true isentropic pressure-volume relation: (1) by the tangent at the point A corresponding to stagnation conditions; (2) by the tangent at the point B corresponding to conditions at infinity; (3) by the chord AB. All three yield values of the maximum velocity in the flow about a circular cylinder correct to terms in the square M_1^2 of the Mach number at infinity. The Kármán-Tsien variant (2) yields the best approximation to the term in M_1^4 as obtained by I. Imai [Proc. Phys.-Math. Soc. Japan (3) **23**, 180–193 (1941)] by the Rayleigh-Janzen method. Thus for this particular case the accuracy of these variants is comparable to that of the second Rayleigh-Janzen approximation. *J. H. Giese.*

Helfer, H. Lawrence. Waves of finite amplitude in an infinite homogeneous medium. Astrophys. J. **119**, 34–41 (1954).

Jeans's theory of gravitational instability is extended to cover certain kinds of disturbances of finite amplitude. The author investigates plane travelling (or stationary) waves of finite amplitude in an ideal gas moving in its own gravitational field, with all diffusive effects neglected. These waves are analyzed mathematically in a frame of reference in which the wave is stationary. The resulting ordinary differential equation of the second order is solved explicitly in terms of an integral, which the author has evaluated numerically. Three kinds of waves are found to be possible, a stationary wave periodic in space, a travelling wave

periodic in space, and a solitary wave. In all cases the wave lengths are close to the Jeans critical wave length, even for quite large amplitudes. However, the author points out that one can still not conclude that waves of greater length than this would necessarily develop increasing amplitudes, as an ordinary "interchange of stabilities" argument would indicate; for such waves might well develop shock waves inside them, and the energy dissipation in those shock waves might control, or even reverse, the process of amplitude increase.

M. J. Lighthill (Manchester).

Chester, William. The shock strength in the regular reflection of weak shock waves. J. Aeronaut. Sci. **21**, 347–349 (1954).

Gilbarg, D., and Shiffman, M. On bodies achieving extreme values of the critical Mach number. I. J. Rational Mech. Anal. **3**, 209–230 (1954).

Voici une nouvelle extension des méthodes de comparaison qui ont été utilisées avec succès par Gilbarg [même J. **1**, 309–320, 411–417 (1952); **2**, 233–251 (1953); ces Rev. **13**, 877; **14**, 279, 920], Shiffman [ibid. **1**, 605–652 (1952); ces Rev. **14**, 510], et J. Serrin [ibid. **1**, 1–48 (1952); Amer. J. Math. **74**, 492–506 (1952); ces Rev. **13**, 877] dans la théorie des écoulements incompressibles et des écoulements subsoniques.

Sont considérés ici les écoulements symétriques plans, uniformes à l'infini, pour lesquels la vitesse du son peut être atteinte, mais non dépassée, en certains points et le théorème fondamental de comparaison établi pour ces mouvements s'énonce ainsi: "Soient B et $\bar{B}$ des profils symétriques tels que $\bar{B}\subset B$ et supposons que B et $\bar{B}$ aient un point commun de contact P régulier. Si deux écoulements du type considéré autour de B et $\bar{B}$ ont des vitesses à l'infini q_0 et $\bar{q}_0$ avec $\bar{q}_0 \geqq q_0$, alors les vitesses en P satisfont à la relation

$$\bar{q}(P) \geqq q(P)$$

et, de plus, l'égalité n'a lieu que lorsque les deux écoulements sont identiques. En particulier, la vitesse $q(P)$ ne peut être sonique que si les écoulements sont identiques." Ce théorème repose lui-même sur le lemme suivant: "Si un écoulement plan, symétrique, autour d'un obstacle est subsonique à l'infini et n'est nulle part supersonique, le maximum de la vitesse est alors atteint sur le profil de l'obstacle."

Ces résultats sont utilisés pour traiter les deux problèmes d'extrêmum ci-après, qui sont en relation avec la détermination des profils optima d'aile portante. 1. Parmi tous les profils ayant un rapport donné entre les différences des abscisses et des ordonnées extrêmes, déterminer celui pour lequel le nombre de Mach critique est maximum (le nombre de Mach critique est le nombre de Mach à l'infini pour lequel l'écoulement devient sonique en des points du profil). 2. Parmi tous les profils ayant une largeur et une courbure maximum données, définir celui pour lequel le nombre de Mach critique est minimum. Le profil extrêmal du premier problème est constitué par deux segments verticaux, reliés par deux arcs convexes sur lesquels la vitesse est sonique en tout point. La détermination de ces arcs à vitesse sonique est effectuée par la méthode de l'hodographe. La solution du deuxième problème est l'arc circulaire dont la largeur et la courbure ont les valeurs fixées. Enfin, pour terminer, les auteurs montrent que les considérations précédentes s'appliquent aux écoulements à symétrie de révolution.

R. Gerber (Toulon).

Whitehead, L. G. Two-dimensional wind-tunnel interference. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2802** (1950), 15 pp. (1954).

Inviscid, irrotational potential flow is considered, so that the theory derived may be considered relevant to wind tunnels operating at low subsonic speeds. The author considers flow about a body in the tunnel when the sides are rigid and straight, when they are isobars (loci of constant pressure as in the case of the boundaries of a free jet), and when they are rigid except near the body, where they are isobars. One body inserted in the flow is a nearly circular oval; another is shaped roughly like an elongated ellipse, but is designed to experience constant pressure over most of its surface. The bulk of the paper is devoted to the by no means trivial problem of the development and study of the necessary conformal transformations. The inserted bodies have been chosen carefully so that most results are expressed in terms of elementary or elliptic functions.

E. Pinney (Berkeley, Calif.).

Jones, W. P. Supersonic theory for oscillating wings of any plan form. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2655** (1948), 11 pp. (1953).

This is a theoretical analysis of the flow around an oscillating wing of arbitrary plan form, according to the linearised theory of supersonic flow. A Green's function method is used and the solution for the definitely supersonic case is written down explicitly. For the case of partly subsonic leading edges, an integral equation is derived for the velocity potential. The two-dimensional case and the case of steady motion are considered as special examples. It is to be noted that the present paper was presented already in 1948, and it must be pointed out that the usefulness of the reports issued in this series could be enhanced considerably through speedier publication. *A. Robinson*.

Lawrence, H. R., and Flax, A. H. Wing-body interference at subsonic and supersonic speeds—survey and new developments. J. Aeronaut. Sci. **21**, 289–324, 328 (1954).

Roberts, Richard C., and Riley, James D. A guide to the use of the M.I.T. cone tables. J. Aeronaut. Sci. **21**, 336–342 (1954).

Fjørtoft, Ragnar. On the changes in the spectral distribution of kinetic energy for twodimensional, nondivergent flow. Tellus **5**, 225–230 (1953).

Two-dimensional motion of a fluid on the surface of a sphere is considered, with neglect of viscosity in the first instance. The author's equations of motion (1) imply that the fluid is of constant density; his stream-function ψ is defined as a function of the space-coordinates but his equation (5) implies that it is also an explicit function of the time. The total kinetic energy and the total vorticity squared are shown to be integral constants of the motion. By expanding ψ as a series of eigenfunctions, the time-changes of the spectral distribution of kinetic energy are determined. Only a fraction of the initial energy flows into motions of smaller scale, a greater fraction must flow into larger-scale motions. Stability is also considered and it is shown that a stability proof pre-supposes that both kinetic energy and vorticity are small. The effect of viscosity is also briefly discussed. The text of the paper contains many misprinted subscripts.

G. C. McVittie (Urbana, Ill.).

Bolin, Bert. The adjustment of a non-balanced velocity field towards geostrophic equilibrium in a stratified fluid. Tellus 5, 373–385 (1953).

An incompressible fluid on a rotating earth, having a small vertical density-gradient, is moving eastwards with uniform speed. The adjustment of this flow to geostrophic motion is discussed by a perturbation method. The process depends on the width of the current, on the velocity profile in the vertical and on the vertical density-gradient. Vertical variations in the unbalanced motion will set up internal oscillations of considerably larger vertical amplitude than occur in the adjustment of the mean motion. In the final equilibrium state the mean motion of the fluid is much the same as before adjustment, and the vertical gradients of the velocity field will have been smoothed out, the more so the broader the original current. *G. C. McVittie.*

Truesdell, C. Precise theory of the absorption and dispersion of forced plane infinitesimal waves according to the Navier-Stokes equations. J. Rational Mech. Anal. 2, 643–741 (1953).

Considérons un milieu continu en mouvement à une seule dimension, par rapport à un plan de référence $x=0$ et suivant la loi $I=I_0\Re e^{i\omega t}$, où I peut représenter différentes variables: déplacement d'une particule, vitesse, accélération, pression, densité ou température. Imprimons à ce mouvement une petite perturbation de façon à pouvoir négliger dans les équations différentielles du mouvement les termes non linéaires. Supposons que le mouvement actuel pourrait être représenté comme une propagation oscillatoire amortie sous la forme: $I=I_0\Re e^{\chi x+i\omega t}$ où χ est une quantité complexe. l_0 étant la longueur d'onde correspondante par rapport à une certaine vitesse fixe V_0 et σ une quantité sans dimension: $\sigma=l_0\chi$, le rapport $r=(V/V_0)^2$ (où V est la vitesse actuelle) est la mesure de la dispersion, le coefficient d'absorption par longueur d'onde étant: $A=l|\Re\chi|$. Avec ces définitions la formule devient:

$$(1)\qquad I=I_0\exp\left[\pm Ax/l\right]\cos\omega(\pm x/V+t).$$

L'auteur introduit, en outre, deux nombres sans dimensions: le nombre thermo-visqueux: $Y=\kappa/(\lambda+2\mu)c_p$ et le nombre de fréquence: $X=(\lambda+2\mu)\omega/\rho V_0^2$ où λ et μ sont les deux viscosités, κ le coefficient de conductibilité calorifique, γ le rapport des chaleurs spécifiques. On voit que toutes les mesures d'absorption et de dispersion peuvent être exprimées en fonction de X, Y et γ.

Une fois ces notations introduites, l'auteur nous présente un exposé historique et critique très détaillé du développement de la théorie de la dispersion et de l'absorption, en commençant par les deux mémoires fondamentaux de Stokes et en terminant par les toutes récentes recherches modernes.

La deuxième partie du mémoire est consacrée à l'étude détaillée de l'équation de Kirchhoff-Langevin. L'auteur considère les équations linéarisées du mouvement, de la continuité et de l'énergie. Pour que le système possède des solutions de la forme (1) il faut que le paramètre σ satisfasse à une équation caractéristique, qui se réduit à une équation biquadratique en σ. Pour le cas barotrope cette équation se réduit à un produit de deux facteurs quadratiques en σ

$$\left[-i+(X-i)\sigma^2/4\pi\right]\left[i-XY\sigma^2/4\pi^2\right]=0$$

dont le premier se rapporte aux ondes de pression et le second aux ondes thermiques. Il en résulte, dans ce cas, qu'aux basses fréquences les ondes thermiques sont absorbées relativement beaucoup plus que les ondes de pression, tandis que lorsque les fréquences croissent, la situation se renverse. On en déduit d'autres conséquences intéressantes pour différentes valeurs de X et Y.

Dans le cas général on peut, par un changement approprié des variables, ramener l'équation à la forme:

$$(2)\qquad -2(2\pi/\sigma)^2=1+iKQ+z^{1/2}$$

où K et Q sont fonctions de X et Y, et z la variable complexe $x+iy$ (on pose $\gamma Y\neq 1$). En étudiant les solutions de (2) l'auteur établit une classification des fluides, en distinguant trois types principaux: bons, moyens et faibles conducteurs. Entre ces principaux types il y a lieu de distinguer différents sous-types. Suivant la double détermination de la racine carrée de la variable complexe z, il y a lieu de distinguer deux types d'ondes: onde type I et onde type II, suivant la dénomination de l'auteur. Vers la fin de son mémoire l'auteur fait une étude analytique très complète de la courbe (2), suivant les différentes valeurs de X, Y et γ.

Le mémoire est suivi de tables en vue des diverses applications numériques. *M. Kiveliovitch* (Paris).

Knudsen, John R. The effects of viscosity and heat conductivity on the transmission of plane sound pulses. J. Acoust. Soc. Amer. 26, 51–57 (1954).

The author presents a theory of the effect of viscosity and heat conduction on the propagation of plane sound pulses of infinitesimal amplitude through an ideal gas. The behaviour of such pulses is, of course, well known. The wave form suffers simple diffusion with a diffusivity $4\nu/3+(\gamma-1)\kappa$, where ν is the kinematic viscosity, γ the ratio of the specific heats, and κ the thermal diffusivity. The author arrives at a result of this form after seven pages, but his diffusivity is different. The only errors which the reviewer has noted are that the viscous terms in the Navier-Stokes equations are written down incorrectly, and that the specific heat of the gas has been tacitly taken as unity; in addition, a special value of the Prandtl number has been used; but by themselves these points do not explain the discrepancy.

M. J. Lighthill (Manchester).

Keller, Joseph B. Decay of spherical sound pulses due to viscosity and heat conduction. J. Acoust. Soc. Amer. 26, 58 (1954).

The results of the preceding paper, errors and all, are extended to spherical pulses. The author appears surprised that his conclusions cannot be applied to the decay of spherical shocks at large distances from the centre of an explosion, for which they would predict amplitude decay like $r^{-3/2}$, where r is distance from the centre. But it is, of course, well known that all shock waves are essentially non-linear phenomena in which a balance between the diffusion (linear) and convection (non-linear) of sound is maintained; and also that the decay of motions involving shock waves can be studied by means of an idealized representation in which the shock wave is replaced by a discontinuity. This is because effectively all the energy dissipation occurs inside the shock wave. The amplitude decay actually follows an $r^{-1}[\log(r/r_0)]^{-1/2}$ law asymptotically. This has been known on physical grounds for many years [for a mathematical investigation see Whitham, Proc. Roy. Soc. London. Ser. A. 203, 571–581 (1950); these Rev. 13, 180].

M. J. Lighthill (Manchester).

Embleton, T. F. W. Mean force on a sphere in a spherical sound field. I. Theoretical. J. Acoust. Soc. Amer. **26**, 40–45 (1954).

The theory is developed on the lines laid down in an earlier paper by L. V. King [Proc. Roy. Soc. London. Ser. A. **147**, 212–240 (1934)]. Only the first-order term in the expression of the pressure variation in the propagation medium (air) is taken into account by substituting the velocity potential expression into the equation for the pressure variation. The boundary condition that the medium is at all points in contact with the rigid sphere is used. The velocity potential is expressed as a series of spherical harmonic functions and the mean values are then calculated. Numerical results are given in the form of curves showing the radiation force at a distance r from the center of the spherical sound field over the radius a of the obstacle. *M. J. O. Strutt.*

Elasticity, Plasticity

Signorini, Antonio. Sopra un'estensione della teoria linearizzata dell'elasticità. Univ. e Politecnico Torino. Rend. Sem. Mat. **12**, 83–93 (1953).

The author formulates a linearized theory of elasticity based on a quadratic form in nine variables. In this theory he obtains an identity from which follow strict analogies of the theorems of minimum energy, reciprocity, etc. in the classical linear theory based on a form in six variables. *C. Truesdell* (Bloomington, Ind.).

Signorini, A. Über eine Erweiterung der linearisierten Theorie der Elastizität. Österreich. Ing.-Arch. **8**, 47–53 (1954).

Except for omission of some remarks at the end on the factor of safety, a translation of the lecture reviewed above. *C. Truesdell* (Bloomington, Ind.).

Moisil, Ana. Les relations entre les tensions pour les corps élastiques à isotropie transverse. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3** (1951), 473–480 (1952). (Romanian. Russian and French summaries)

For a body with transverse isotropy in the classical linear theory of elasticity, the author obtains the explicit form of the appropriate generalization of the Beltrami differential equations for the stress components. *C. Truesdell.*

Adkins, J. E. Some generalizations of the shear problem for isotropic incompressible materials. Proc. Cambridge Philos. Soc. **50**, 334–345 (1954).

In the classical theory of finite elastic strain of incompressible bodies, the author considers the class of deformations in which the point x_1, x_2, x_3 is carried into

$$\lambda x_1,\ \lambda x_2,\ x_3/\lambda^2+f(\lambda x_1, \lambda x_2).$$

He reduces the problem to solution of a pair of complicated differential equations connecting the strain-energy of the material, the hydrostatic pressure, and f. From this point on he considers only Mooney's form of the strain-energy. For this case one of the equations reduces to $\nabla^2 f=\text{const.}$, while the other is satisfied identically. The author also constructs a formulation in complex variables, thus expressing all stresses and displacements in terms of a single arbitrary analytic function and its derivatives. He works out the application of conformal mapping to the problem and illustrates the results by some special cases. One of these is plane shear superposed on finite extension, already treated by the author, Green, and Shield [Philos. Trans. Roy. Soc. London. Ser. A. **246**, 181–213 (1953); these Rev. **15**, 369]. The second is shear of a cylindrical annulus. He discusses the solution for a material sheared between rigid cylinders, to which it adheres, using as illustrations the cases when the generating curves are confocal ellipses or concentric circles. *C. Truesdell* (Bloomington, Ind.).

Tremmel, E. Zur Theorie kreisberandeter Bogenscheiben. Österreich. Ing.-Arch. **8**, 11–38 (1954).

The solution in bi-polar co-ordinates is given of certain dislocational plane strain (or stress) problems for a cross-section bounded by two non-concentric circles and with a radial cut at the thickest section. Stress boundary conditions only are considered; these involve forces applied to the radial cuts and to the outer boundary. The solutions obtained are used to develop an approximate treatment of the engineering problem of an arch bridge under uniform load, and the results obtained are compared with those based upon elementary curved-beam theory. The author's analysis involves the use of stress functions. In the reviewer's opinion Muskhelishvili's complex variable method could be employed with advantage. *H. G. Hopkins.*

Arf, C. Sur un problème de frontière libre d'élasticité bidimensionelle. Bull. Tech. Univ. Istanbul **5** (1952), 13–16 (1953). (Turkish summary)

In a previous note [same Bull. **4**, no. 1, 1–4 (1952); these Rev. **14**, 924] the author established an analogy between the free stream lines in two-dimensional potential flow and free boundaries in two-dimensional elasticity having constant hoop stress. The object of this note is to show that there exists another distribution of stress which also gives constant hoop stress along the same boundaries but which does not necessitate a load distribution along the remaining part of the boundary as in the previous paper. Such a distribution of stress is found by means of general results given in a communication to the 6th International Congress of Applied Mechanics, and by means of these general results the problem is reduced to the solution of $G(t)$ when it satisfies the conditions

$$\operatorname{Im} G(t)=0 \quad \text{when} \quad |t|\leq 1;$$
$$\operatorname{Im} G(t)=1-[t+(t^2-1)^{1/2}]^{-2} \quad \text{when} \quad |t|>1$$

and this function is easily found. *R. M. Morris.*

Birman, S. E. On the plane problem of bodies composed of parts with different elastic constants. Doklady Akad. Nauk SSSR (N.S.) **93**, 989–992 (1953). (Russian)

The author's earlier results [same Doklady (N.S.) **62**, 187–190 (1948); these Rev. **10**, 341] are used to solve the problem of elastic deformation of a composite infinite strip consisting of two infinite strips loaded along the line of contact. Several special cases of loading are considered. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Fichera, Gaetano. Sulla torsione elastica dei prismi cavi. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12**, 163–176 (1953).

In the classical linear theory of elasticity, the author considers the torsion of a prism whose cross-section is the region between two polygons. It has long been known that at a reentrant corner of angle μ the warping function has the form

$$\phi=q[\phi]\rho^{1/\mu}\cos\frac{\theta}{\mu}+\phi_0,$$

where ρ, θ are polar coordinates with origin at the corner, $q[\phi]$ is a numerical coefficient, and ϕ_0 is a function whose first derivatives are continuous near the corner. The value of $q[\phi]$ could not in general be estimated, nor could it be shown even that $q\neq 0$. The author constructs majorant and minorant sequences $q_k'[\phi]$ and $q_k''[\phi]$, both convergent to $q[\phi]$. His method rests upon use of two harmonic functions, u and v. The former reduces to x^2+y^2 on the entire boundary; the latter vanishes on the inner polygon and has the value 1 on the outer. The author reduces the problem of determining majorant and minorant sequences convergent to $q[\phi]$ to analogous problems for $q[u]$, $q[v]$, and

$$\alpha = \int_C (\partial u/\partial \nu)ds \bigg/ \int_C (\partial v/\partial \nu)ds,$$

where C is any curve in the cross-section and inclosing the inner polygon. These three problems he solves by Hilbert space methods, using expansions in terms of the set $\Re z^h$, $\Im z^h$, $h=0, \pm 1, \pm 2, \cdots$. The main tool is furnished by expressions for $q[u]$ and $q[v]$ as definite integrals. For the case of two homothetic squares, one having sides of length twice the other's, the author asserts that at an internal vertex $q[\phi]=2.14\cdots$. *C. Truesdell.*

Išlinskiĭ, A. Yu. On an integro-differential relation in the theory of an elastic cord (cable) of variable length. Ukrain. Mat. Žurnal **5**, 370–374 (1953). (Russian)

The upper end of an elastic cable is attached to a drum, which is forced to rotate, and its lower end supports a heavy mass, m. The displacement of a cross-section of the cable is assumed to be of the form $u(x, t)=x\phi(t)$, and an approximate equation for $\phi(t)$ is found, essentially by Rayleigh's method. To illustrate the degree of approximation, if the cylinder is held fixed, the resulting equation leads to the familiar correction wherein one-third the spring mass is added to the spring-supported mass to find the effective mass. *R. E. Gaskell* (Seattle, Wash.).

Kalandiya, A. I. Bending of an elastic plate in the form of an elliptic ring. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 693–704 (1953). (Russian)

The problem of transverse deflection of a thin elastic plate in the form of a confocal elliptic ring, with one boundary clamped and the other free, is solved by determining the appropriate analytic functions of a complex variable. The determination of these functions calls for the solution of an infinite system of linear algebraic equations, and a method of solving them is provided. As a special case the author deduces a solution of the fundamental biharmonic boundary-value problem for the confocal elliptical ring, treated approximately by M. P. Šeremet'ev [same journal **17**, 107–113 (1953); these Rev. **14**, 1145]. The case of uniform loading of the plate is considered as an example.
I. S. Sokolnikoff (Los Angeles, Calif.).

Shaw, F. S., and Perrone, N. A numerical solution for the nonlinear deflection of membranes. J. Appl. Mech. **21**, 117–128 (1954).

Parkus, Henry. Thermal stress in pipes. J. Appl. Mech. **20**, 485–488 (1953).

Temperature distribution and thermal stress in a semi-infinite hollow cylinder are considered for the case where an incompressible hot liquid, flowing inside the cylinder under steady-state conditions, is transferring heat to the outside wall of the cylinder. If the inner radius is a, the outer radius is b, and the average temperature of incoming fluid entering at station $z=0$ is θ, the boundary-value problem for the temperature distribution $T(r, z)$ is formulated as:

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \quad \text{for } z>0 \text{ and } a<r<b;$$

$H\partial T/\partial r - \partial T/\partial z = 0$ at $r=a$, $z>0$; $T=0$ at $r=b$, $z>0$; $T=\theta f(r)$ at $z=0$, $a\leqq r\leqq b$ with $f(a)=1$, $f(b)=0$. Here H and f are a given constant and a given function respectively. The thermal stresses corresponding to this temperature distribution are then found by determining the appropriate stress functions [S. Timoshenko and J. N. Goodier, Theory of elasticity, 2nd ed., McGraw-Hill, New York, 1951, p. 433, also p. 343; these Rev. **13**, 599]. The solutions are presented for both temperature and stress problems as formal expansions in infinite series of eigenfunctions; no discussion of the results or numerical applications are given.
W. Nachbar (Seattle, Wash.).

Funaioli, Ettore. Sullo slittamento elastico nel rotolamento. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 15–24 (1953).

An elastic circular cylinder is pressed against, and is constrained to roll on, the plane surface of an elastic body. Rough contact between the bodies is assumed. This paper discusses, through a plane strain analysis, the question of slipping over the contact region for cases when the coefficient of dynamic friction is either constant or proportional to a power of the velocity. A numerical example is given.
H. G. Hopkins (Providence, R. I.).

Masur, E. F. Lower and upper bounds to the ultimate loads of buckled redundant trusses. Quart. Appl. Math. **11**, 385–392 (1954).

Two principles are established, relevant to lower and upper bounds to the ultimate loads of buckled, redundant, rigid-jointed trusses, under the assumption of elastic deformations. The first principle permits the determination of a sequence of lower bounds which converges to the true value.
F. B. Hildebrand (Cambridge, Mass.).

Thomas, T. Y. Determination of the plastic yield condition as a variational problem. Proc. Nat. Acad. Sci. U. S. A. **40**, 322–331 (1954).

The author starts from the hypothesis that plastic flow occurs in such a manner as to render stationary some energy integral extended over the region of flow. Without referring to the known variational principles of the mathematical theory of plasticity, which can be interpreted in precisely these terms, he then investigates the effect, on the plastic flow, of the requirement that the integral of some differentiable invariant function ψ of the deviations of stress and velocity strain has a stationary value over the plastic domain. The assumption is introduced that the stress deviation is a differentiable tensor invariant of the deviation of the velocity strain. [In the light of experimental evidence, this assumption of differentiability has been abandoned in several recent investigations of the general structure of the plastic stress-strain relations; see, for instance, papers presented by Budiansky, Dow, Peters, and Shephard [Proc. 1st U. S. Nat. Congress Appl. Mech., Chicago, 1951, Amer. Soc. Mech. Engrs., New York, 1952, pp. 503–512] and Sanders at the Second Congress in 1954 (to appear). Koiter [Quart. Appl. Math. **11**, 350–354 (1953); these Rev. **15**, 583] has shown that the variational principles of plasticity can be extended to include stress-strain relations of this

kind.] When the conditions for the stationary character of the integral of ψ over the plastic region are combined with the equations of equilibrium and the condition of incompressibility, there results an overdetermined system. This overdetermination is found to disappear only if the function ψ reduces to a constant as a function of the components of the deviation of the velocity strain. This result has the nature of a yield condition. *W. Prager.*

Wang, Alexander J., and Prager, William. Thermal and creep effects in work-hardening elastic-plastic solids. J. Aeronaut. Sci. 21, 343–344, 360 (1954).

The authors derive two extremum principles for work hardening elastic-plastic solids, taking into account thermal and creep effects. These principles are generalizations of those previously derived by P. Hodge and W. Prager [J. Math. Physics 27, 1–10 (1948); these Rev. 10, 83] and R. Hill [The mathematical theory of plasticity, Oxford, 1950; these Rev. 12, 303]. *J. L. Ericksen.*

Hoff, N. J. Approximate analysis of structures in the presence of moderately large creep deformations. Quart. Appl. Math. 12, 49–55 (1954).

Under the assumption that primary creep deformations and elastic deformations can be neglected relative to secondary creep deformations, when creep strains of the order of one or two percent are developed during the lifetime of a structure, it is shown that the stress distribution corresponding to one of a rather general class of non-linear creep laws is the same as that corresponding to the solution of a certain analogous perfectly-elastic problem. A simple example is presented to illustrate the calculation, and the validity of the basic assumption is investigated in that case. *F. B. Hildebrand* (Cambridge, Mass.).

Koço, Petrika. Sur l'équilibre d'une classe de corps viscoélastiques. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 3, 227–243 (1951). (Romanian. Russian and French summaries)

The author studies the Meyer-Voigt theory of infinitesimal visco-elastic deformation of incompressible bodies. Using a matrix method due to Gr. C. Moisil, she derives various differential equations which follow from the basic equations of this theory. Her results may be obtained without any calculation if we simply notice that all the equations of the usual theory of elasticity remain valid if we replace the displacement vector **u** by $\mathbf{u}+\tau\partial\mathbf{u}/\partial t$, where τ is a material constant. Hence, for example, Beltrami's compatibility equations for the stress field are unaltered. *C. Truesdell* (Bloomington, Ind.).

MATHEMATICAL PHYSICS

Rose, M. E. Spherical tensors in physics. Proc. Phys. Soc. Sect. A. 67, 239–247 (1954).

A vector field with cartesian coordinates A_i is said to determine the spherical vector $T_1^M(A_i)=y_1^M(A_i)$ where y_1^M are the three solid harmonics of degree one. Spherical tensors of higher rank are defined in terms of products of spherical vectors in accordance with the Clebsch-Gordon formula. Applications to the theory of beta-transitions, gamma ray emission, angular correlation, and the static interaction of a multipole with a surrounding spin system or field are considered. *A. H. Taub* (Urbana, Ill.).

Bastin, E. W., and Kilmister, C. W. The concept of order. I. The space-time structure. Proc. Cambridge Philos. Soc. 50, 278–286 (1954).

The authors expound an unorthodox view of the relation of mathematics to physics. They define an algebraic structure which is claimed to have a property of fundamental physical importance and which is such that from it more complex structures may be obtained. The authors further state that at every stage the physical meaning of every structure in the theory can be stated. The main part of this paper is concerned with a set, whose only postulate is closure under one binary operation, and certain of its subsets. The physical meaning of the elements of the set and the binary operation is not completely clear to the reviewer. *A. H. Taub* (Urbana, Ill.).

Electromagnetic Theory

✶Ferraro, V. C. A. Electromagnetic theory. The Athlone Press, London, 1954. viii+555 pp. $7.00.

The work under consideration is a treatment of electricity and magnetism at an intermediate level. It is addressed to honor students in mathematics and sets a standard of rigor a bit higher than found in books written for students of physics. The selection of material is essentially classical and except for brief treatments of the rectangular wave guide and the interaction of charge and field, one might suppose that the book had been written at the turn of the century. Within the bounds set by the author, ones not suggested by the title, the material is carefully and fully covered. The exposition is lucid and the pace quite uniform. The reviewer notes only the following exceptions.

1.) The treatment of charge layers seems inadequate. The discontinuity of the electric vector at the simple layer is not discussed and the dipole layer is mentioned only in its connection to magnetostatics. It would be preferable to include all such material in the chapter devoted to continuous distributions.

2.) The infinity condition required for the proof of the uniqueness theorem for the potential equation is used but not explicitly stated. This is unfortunate from the standpoint of exactness and in that the asymptotic behavior of the solution is an extremely revealing characteristic of a partial differential equation. It is only when the student has become aware that the Laplace equation exhibits solutions which vanish as any power of $1/r$ at infinity that the behavior of the Helmholz equation comes as a surprise.

3.) As is to be expected, the existence theorem for the Dirichlet problem is not proved, but it is stated on page 146 with insufficient restrictions to avoid the counterexample given by Lebesgue [C. R. Soc. Math. France 1913, 17].

4.) Section 257 seems confused. If one is given Maxwell's equations and the right to interchange the time and spatial derivatives, the equation of continuity may be made to follow. Alternately given the latter plus Ampere's law for time-dependent phenomena, etc., one can derive Maxwell's equation. The development of the differential form of the equation of continuity from the assumption of its existence in integral form is a somewhat unrevealing procedure.

5.) The treatment of the method of images fails to suggest adequately to the student the rocky road of convergence problems which lies just beyond the textbook examples.

6.) The references and acknowledgment of source, primarily British, is inadequate for a book at the senior level.

7.) Without a preliminary discussion, e.s.u., e.m.u. and Gaussian units are used in turn. If the author chose to avoid the m.k.s. system, that was his privilege. He should then have planned to standardize, in the beginning, as he would be forced to at the end, on a single system and avoided the confusion inherent in the introduction of three. Historically, the student has had to suffer confusion in this matter, but it is held by some that a purpose of the study of history is to avoid the repetition of mistakes. For the author this could have been easily done, for the arrangement of the topics is such that the problem does not arise until page 406. There he would have merely to resist the temptation to introduce a second definition of current.

With the exception of the last, these are all minor points and ones easily amended by the lecturer. Subject to the reservation as to units, the book is to be recommended as an intermediate text in the static and quasistatic electric and magnetic phenomena. *W. K. Saunders.*

Slansky, Serge. Sur le tenseur de Maxwell. C. R. Acad. Sci. Paris **238**, 1103–1104 (1954).

The author adds to the usual stress-energy tensor a divergence of a third-order tensor which is anti-symmetric in two of the indices, and obtains a modified stress-energy tensor of the Maxwell field which has the same divergence as the usual tensor. The additional terms depend on the field variables and the vector potential. It is stated that the new stress-energy tensor has the following two properties. (1) For a plane monochromatic wave the energy density is a constant. (2) In the case of a static field the energy density vanishes in the regions where there is no current vector. The stress-energy tensor given by the author is gauge dependent but, as the author points out, the Lorentz-force equations will be gauge independent. *A. H. Taub* (Urbana, Ill.).

Fejer, J. A. The diffraction of waves in passing through an irregular refracting medium. Proc. Roy. Soc. London. Ser. A. **220**, 455–471 (1953).

In this paper the diffraction of a plane wave by a sheet which imposes on the incident wave irregular and random variations of amplitude and phase in two-dimensions is considered. The relation

$$(1)\quad \rho(\xi,\eta)=\text{const.}\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|N(s_1,s_2)|^2 \times\exp\{ik(s_1\xi+s_2\eta)\}ds_1ds_2,$$

and its inverse are established between the angular power spectrum $|N(s_1,s_2)|^2$ of the diffracted beam and the auto-correlation function $\rho(\xi,\eta)$ of the Hertzian vector on the diffracting screen. In (1) s_1, s_2 define the directions of the wave normal and k is the wave number. Similarly, by scattering by a volume distribution of irregularities, the relation between the power, P, scattered per unit solid angle, per unit volume and per unit incident power density and the auto-correlation function $\rho(\xi,\eta,\zeta)$ of the fluctuation in the dielectric constant, ϵ, is given by

$$(2)\quad P=\sin^2\gamma\frac{\pi^2[(\Delta\epsilon)^2]_{\text{av}}}{\lambda^4\epsilon^2}\int\int\int\rho(\xi,\eta,\zeta) \times\exp\{-ik[(s_1-S_1)\xi+(s_2-S_2)\eta+(s_3-S_3)\zeta]\}d\xi d\eta d\zeta,$$

where γ is the angle between the directions of the scattered wave and the electric field of the wave incident in the direction S_1, S_2, S_3, and λ is the wave length. For an isotropic distribution of the irregularities, (2) reduces to

$$(3)\quad P=\frac{\pi^2[(\Delta\epsilon)^2]_{\text{av}}\sin^2\gamma}{2\lambda^3\epsilon^2\sin\frac{1}{2}\Phi}\int_{-\infty}^{+\infty}r\rho(r)\sin[2kr\sin\tfrac{1}{2}\Phi]dr,$$

where $s=2\sin\frac{1}{2}\Phi$.

It is also shown that the effect of multiple small angle scattering can be allowed for by superposing the effects of power densities Q_n $(n=0,1,\cdots)$ resulting from waves scattered n times and that $Q_n=Q_0e^{-B_0}B_0^n/n!$ where B_0 is the "optical thickness" of the screen. The various formulae developed in the paper are illustrated by considering a number of special cases. *S. Chandrasekhar.*

Ferraro, V. C. A. On the reflection and refraction of Alfvén waves. Astrophys. J. **119**, 393–406 (1954).

It is known that in the presence of a uniform magnetic field of intensity H_0, waves with a velocity $V=H_0/(4\pi\rho)^{1/2}$ can be propagated in an incompressible fluid of infinite conductivity and density ρ. In this paper the author considers the reflection and refraction of these waves at the interface of two infinite media of densities ρ_1 and ρ_2 separated by a surface $z=0$, the densities ρ_1 and ρ_2 prevailing for $z<0$ and $z>0$, respectively. The magnetic fields of the waves in the two media can be written in the forms

$$(1)\quad \begin{aligned}\mathbf{h}_1=&\mathbf{A}(x,\,-my+lz)\exp\left\{i\omega\left(t-\frac{ly+mz}{V_1}\right)\right\}\\ &+\mathbf{B}(x,\,-my+lz)\exp\left\{i\omega\left(t+\frac{ly+mz}{V_1}\right)\right\}\end{aligned}$$

and

$$(2)\quad \mathbf{h}_2=\mathbf{C}(x,\,-my+lz)\exp\left\{i\omega\left(t-\frac{ly+mz}{V_2}\right)\right\},$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are arbitrary functions of arguments specified, ω is the circular frequency of waves, $V_1=H_0/(4\pi\rho_1)^{1/2}$, $V_2=H_0/(4\pi\rho_2)^{1/2}$ and $(0,l,m)=(0,\cos\beta,\sin\beta)$ specifies the direction of motion of the waves, β denoting the inclination of the direction of H_0 to the z-axis. In (1) the first and the second terms represent the incident and the reflected waves, respectively, while (2) represents the refracted wave. The continuity of the electric and the magnetic fields as well as the normal component of the velocity at the interface require that $\mathbf{h}_1=\mathbf{h}_2$ and $\mathbf{v}_1=\mathbf{v}_2$ (except when H_0 is parallel to the surface $z=0$). (The velocity $\mathbf{v}=\pm\mathbf{h}/(4\pi\rho)^{1/2}$, the plus sign to be used for the incident and the refracted waves, while the negative sign is to be used for the reflected wave.) Using these conditions and expressing $\mathbf{A}$ as a harmonic function

$$\mathbf{A}(x,\,-my+lz)=\mathbf{A}_0\exp\left\{\frac{i}{V_1}[-kx+\omega_0(-my+lz)]\right\}$$

where k and ω_0 are constants and $\mathbf{A}_0$ is a constant vector, the author shows that the solutions can be written in the forms

$$\mathbf{h}_1=\mathbf{A}_0\exp\left\{i\left(\omega t-\frac{\mathbf{u}_0\cdot\mathbf{r}}{V_1}\right)\right\}+\mathbf{A}_0'\exp\left\{i\left(\omega t-\frac{\mathbf{u}_1\cdot\mathbf{r}}{V_1}\right)\right\},$$

$$\mathbf{h}_2=\mathbf{A}_0''\exp\left\{i\left(\omega t-\frac{\mathbf{u}_2\cdot\mathbf{r}}{V_1}\right)\right\},$$

where

$$\mathbf{u}_0=[k,\ l\omega+m\omega_0,\ m\omega-l\omega_0],$$

$$\mathbf{u}_1=\left[k,\ l\omega+m\omega_0,\ -l\omega_0-(1+l^2)\frac{\omega}{m}\right],$$

$$\mathbf{u}_2=\left[k,\ l\omega+m\omega_0,\ -l\omega_0-\left(l^2-\frac{V_1}{V_2}\right)\frac{\omega}{m}\right],$$

$$\mathbf{A}_0'=\frac{\rho_2^{1/2}-\rho_1^{1/2}}{\rho_2^{1/2}+\rho_1^{1/2}}\mathbf{A}_0 \quad \text{and} \quad \mathbf{A}_0''=\frac{2\rho_2^{1/2}}{\rho_2^{1/2}+\rho_1^{1/2}}\mathbf{A}_0.$$

From these equations it follows that the only waves that can be reflected or refracted are those in which the magnetic field of the waves (and hence the particle velocity) are parallel to the plane of separation of the two media; also that the laws of reflection and refraction are given by

$$\cot r=\cot i+2\cot\beta\cos\gamma,$$

and

$$\cot R=\frac{V_1}{V_2}\cot i+\left(\frac{V_1}{V_2}-1\right)\cot\beta\cos\gamma,$$

where i, r and R are the angles of incidence, reflection and refraction defined in the usual way and γ is the angle which the common planes of incidence, reflection and refraction make with the plane containing H_0 and the z-axis.

In the second part of this paper the author considers the propagation of hydromagnetic waves in a horizontally stratified isothermal atmosphere (with a density law $\rho_1=\rho_0 e^{-z/z_0}$) overlying a uniform incompressible fluid. In this case the author obtains the solution for the magnetic fields in the two regions in the forms

$$\mathbf{h}_-=\mathbf{A}e^{i\omega t}\cos\left(\frac{\sigma z}{z_0}+\epsilon\right) \quad (z<0)$$

and

$$\mathbf{h}_+=\mathbf{A}e^{i\omega t}[J_0^2(2\sigma)+J_1^2(2\sigma)]^{-1/2}\xi^{1/2}J_1(2\sigma\xi^{1/2}) \quad (z>0),$$

where

$$\sigma=[\omega z_0(4\pi\rho_0)^{1/2}]/H_0=\omega z_0/V_0, \quad \xi=e^{-z/z_0},$$

and J_0 and J_1 are the Bessel function of order zero and one. From the foregoing solution it follows that eventually the magnetic field associated with the waves decreases with height proportional to the density of the atmosphere. The behavior of these solutions is further illustrated.

S. Chandrasekhar (Williams Bay, Wis.).

Villars, F., and Weisskopf, V. F. The scattering of electromagnetic waves by turbulent atmospheric fluctuations. Physical Rev. (2) **94**, 232–240 (1954).

It is shown that under certain assumptions the cross-section, σ, for the scattering by a turbulent atmosphere of electromagnetic waves per unit volume and per unit solid angle in a direction making an angle χ with the direction of incidence is proportional to

$$|M|^2=\left|\int_V d\mathbf{r}\Delta\rho(\mathbf{r})e^{i\mathbf{r}\cdot\mathbf{K}}\right|^2\sin^2\chi,$$

where $\Delta\rho(\mathbf{r})$ denotes the fluctuation in the density in the atmosphere and $|\mathbf{K}|=2k\sin\frac{1}{2}\theta$, k being the wave number of the radiation and θ the scattering angle. It is clear that the quantity whose absolute square is taken in the foregoing expression is simply the value of the spectrum $C(k)$ (as conventionally defined in the theory of turbulence) of the density fluctuation for $k=|\mathbf{K}|$. In the evaluation of $C(\mathbf{K})$ the assumption is made that $\Delta\rho(\mathbf{r})$ is proportional to the fluctuation in the pressure $\Delta p(\mathbf{r})$. The problem therefore reduces to one which has been studied extensively in the literature [Heisenberg, Z. Physik **124**, 628–657 (1948); these Rev. **11**, 63; Batchelor, Proc. Cambridge Philos. Soc. **47**, 359–374 (1951); these Rev. **12**, 874] though the authors rederive the necessary expressions in this connection. The principal result of the calculation is that $|M|^2$ is proportional to $(\sin\frac{1}{2}\theta)^{-11/3}$. *S. Chandrasekhar.*

Twersky, Vic. Certain transmission and reflection theorems. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-54**, i+10 pp. (1953).

The theorems treated here are connected with the scattering and reflection of electromagnetic and acoustic waves. By way of introduction the scattering of a plane wave by a cylinder of arbitrary cross-section and arbitrary constants is considered. The real part of the forward scattered amplitude is proportional to the total "energy cross-section" per unit length of cylinder, which is defined as the power the cylinder dissipates as heat and as scattered radiation per unit length. A corollary of this theorem is applied to obtain an approximate transmission coefficient for a uniform planar distribution of identical cylinders. The analogous reflection problem for an arbitrary cylindrical protuberance on a smooth perfectly reflecting plane is considered and a theorem is derived, stating that the total cross-section is proportional to the real part of the scattering amplitude in the direction of the plane. A corollary of this theorem is then applied to obtain an approximate reflection coefficient for a uniform distribution of identical bosses. *M. J. O. Strutt.*

Twersky, Vic. Multiple scattering of waves by a volume distribution of parallel cylinders. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. **EM-59**, i+16 pp. (1953).

The multiple scattering of a plane electromagnetic or acoustic wave by a uniform distribution of cylinders lying parallel to the z-axis in an infinite slab is considered. The procedure is developed from a previous paper, where the multiple scattering of a plane electromagnetic or acoustic wave by an arbitrary configuration of parallel cylinders was calculated. An average wave function is obtained from the corresponding series expression by integration. Restricting the discussion to an infinite slab of finite thickness, the final scattered wave expression is obtained by further integrations. This leads up to an expression for the average energy flux, which is summed up in the form of a theorem. Special consideration is given to practical applications and especially to grazing incidence. *M. J. O. Strutt* (Zurich).

Zeuli, Tino. Sul problema di Cauchy per la propagazione di onde elettromagnetiche guidate entro un tubo cilindrico circolare. Univ. e Politecnico Torino. Rend. Sem. Mat. **12**, 107–125 (1953).

The author discusses here the propagation of electromagnetic waves in a perfectly conducting cylindrical wave guide filled with a homogeneous dielectric. The waves considered are transverse-magnetic or transverse electric, but are not harmonic waves. The problem reduces to that of solving the wave equation

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}+\frac{\partial^2 U}{\partial z^2}-\frac{\mu\epsilon}{c^2}\frac{\partial^2 U}{\partial t^2}=0$$

under the boundary conditions $U=0$ or $\partial U/\partial r=0$ on $r=a$. What is required is the solution for $t>0$, $z>h$ when the electromagnetic field is initially non-zero and given in a length $0\leqq z\leqq h$ of the tube. The solution in the case of transverse-magnetic waves is of the form

$$U=\sum_{m,j} J_m(k_{mj}r)\cos(m\theta-d_{mj})\Phi_{mj}(z,t),$$

where m is an integer and the constant k_{mj} is chosen so that $J_m(k_{mj}a)=0$. The function $\Phi_{mj}(z,t)$ then satisfies

$$(*)\qquad \frac{\partial^2\Phi}{\partial z^2}-\frac{\mu\epsilon}{c^2}\frac{\partial^2\Phi}{\partial t^2}-k^2_{mj}\Phi=0$$

under initial conditions $\Phi_{mj}=f_{mj}(z)$, $\partial\Phi_{mj}/\partial t=g_{mj}(z)$ when $t=0$, where f_{mj} and g_{mj} are zero outside the interval $0\leqq z\leqq h$. The problem is thus reduced to solving the problem of Cauchy for equation (*), which can be readily done by Riemann's method. The solution so found is finally turned into a form more suited for computation by the use of Graf's addition theorem for the Bessel function. The transverse-electric waves can be discussed in a similar way.

E. T. Copson (St. Andrews).

Tischer, Friedrich J. Zur Fortleitungs- und Anpassungstheorie homogen geführter Wellen. Arch. Elektr. Übertragung **8**, 8–14, 75–84 (1954).

In this neat essay the author emphasizes the fact that the effects of discontinuities and terminating elements on the propagation of electromagnetic waves along a waveguide can be described exclusively in terms of the fields, i.e., without recourse to network theory and transmission line concepts. *C. H. Papas* (Pasadena, Calif.).

Itow, Takeshi. Theoretical treatise on the electromagnetic field in a hollow conductor filled with dissipative mediums. Mem. Fac. Sci. Eng. Waseda Univ. **17**, 15–41 (1953).

Maxwell's field equations are solved for a circular cylindrical wave guide filled with a dissipative medium in terms of products of Bessel functions and the angular trigonometric functions [as shown in J. Stratton, Electromagnetic theory, McGraw-Hill, New York, 1941, p. 360]. It is demonstrated that a double infinity each of transverse electric and transverse magnetic modes exist. The propagation constants are found from the real roots of the nth order Bessel functions of real argument, but are themselves complex. This indicates attenuation and a nonexistence of a sharp cut-off frequency, though an equivalent critical frequency can be defined for each mode, namely

$$V_{n,m}=\frac{\mu_{n,m}}{2\pi a(\mu\epsilon)^{1/2}}$$

where $\mu_{n,m}$ are the nth order roots of $I_n(x)=0$, a the inner radius of the wave guide wall, and μ, ϵ the absolute (Giorgi) permeability and dielectric constant, respectively.

For frequencies very high compared with the critical frequency, the propagation constant approaches that of the unbounded dissipative medium, for the critical frequency, attenuation and phase constant become equal and $\alpha_{n,m}=\beta_{n,m}=(\frac{1}{2}\omega\mu\sigma)^{1/2}$. For frequencies below the critical value, propagation still takes place, but with high attenuation. Plots of field distribution are given for both transverse magnetic and transverse electric waves. The geometric pattern is periodic in the direction of propagation but not symmetrical with respect to the $\lambda/2$ plane as in the lossless dielectric. Extensive computations are also carried through for two dissipative dielectrics in tandem. The patterns of the field distribution are calculated for the case where medium 1 is air and medium 2 is dissipative and for several different phase angles of the wave arriving at the boundary surface.

E. Weber (Brooklyn, N. Y.).

Redheffer, R. M. Novel uses of functional equations. J. Rational Mech. Anal. **3**, 271–279 (1954).

This paper has two parts, dealing respectively with reflection and transmission in stratified dielectrics and with the transfer of energy in a cavity. The author believes that many of the results are new, but the interest centers chiefly in the method, which does not take Maxwell's equations as the point of departure. Part I starts from the law of composition of the complex reflection and transmission coefficients of two plane homogeneous sheets of dielectric, one immediately behind the other. When applied to two sheets of the same material, this yields functional equations for the coefficients $R(x)$, $T(x)$ for a homogeneous sheet of thickness x. These equations are solved, and the method is applied to the discussion of inhomogeneous stratified media. In Part II the basic function is $f(v,v')$, which denotes the power absorbed by an object of volume v in the presence of another object of volume v', both situated in a cavity in which there is a field of radiation, regarded as homogeneous. It is assumed empirically that $f(v,v')+f(v',v)=f(v+v',0)$ and that f is of the form $f(v,v')=va(v')/[v+b(v')]$; the solution $f(v,v')=a_0v/(v+v'+b_0)$ is obtained, a_0 and b_0 being constants. [Misprints in equation (12) and in the Riccati equation on p. 275.] *J. L. Synge* (Dublin).

Durand, Emile. Le champ $\vec{E}$ et l'induction $\vec{D}$ d'une charge électrique ponctuelle dans le vide. C. R. Acad. Sci. Paris **238**, 1478–1480 (1954).

For a single point charge q at the origin a polarization can be defined by

$$\pi(x,y,z)=\frac{q}{4\pi}\int_C\frac{\mathbf{s}}{r}dl,$$

where C is any arbitrary curve starting at the origin and running into infinity, $\mathbf{s}$ a unit vector in the positive tangential direction to C, and r the distance of a point (x,y,z) from the element dl of C. It is shown that this vector potential can be related to the usual scalar potential ϕ of the point charge by $\epsilon_0\phi=-\operatorname{div}\pi$ and interesting explicit forms are given for a few specific choices of the curve C.

E. Weber (Brooklyn, N. Y.).

Power, G. Some perturbed electrostatic fields. Pacific J. Math. **4**, 79–98 (1954).

When a dielectric body is placed in an electrostatic field, the field is perturbed, and the boundary of the dielectric is acted on by mechanical forces due to the refraction of the lines of force. This paper is concerned with finding the resultant mechanical force at a surface of discontinuity between two homogeneous isotropic dielectric media of different specific inductive capacities and the perturbed field for various boundary shapes. In the first section, some two-dimensional problems are discussed, using complex potential functions and complex variable technique. The circular cylinder, elliptic cylinder, parabolic cylinder, circular annulus and slab are considered. In the second section, Stokes' stream function is applied to three-dimensional antisymmetric field problems. *E. T. Copson.*

✶Snow, Chester. Magnetic fields of cylindrical coils and annular coils. National Bureau of Standards Applied Mathematics Series, No. 38, U. S. Government Printing Office, Washington, D. C., 1953. 29 pp. $.25.

A unified method is given for evaluating both components of the magnetic field at any point for axially symmetric coils. The fine structure of the field due to the necessarily helical nature of the windings is ignored, as is also the effect of the axial component of the current. The three classes of coils considered are the cylindrical current sheet, the plane annular current sheet and the multilayer cylindrical coil. The magnetic vector potential is used, from which the magnetic field strength components are obtained by derivation. The integral expression of the vector potential is generalized and expressed by means of complete elliptic integrals. From this the field components are evaluated. The intricate equations are given in a well-ordered form, facilitating their practical application. *M. J. O. Strutt* (Zurich).

Ledinegg, Ernst, und Urban, Paul. Zur Ableitung des Äquivalenzsatzes eines schwach gekoppelten elektromagnetischen Hohlraumsystems. Arch. Elektr. Übertragung **7**, 561–568 (1953).

The general proof of the representation of loosely coupled cavity resonators by equivalent $2n$-pole lumped parameter circuits is based on a perturbation of the first order of the field distribution as obtained from Maxwell's field equations. The frequency dependence of the system can then be represented by the vanishing of a determinant and this can be demonstrated as equivalent to the resolution of the conventional network determinant into partial fractions. The relations are illustrated by detailed analysis of a system of two loosely coupled resonators, one lossless, the other lossy.
E. Weber (Brooklyn, N. Y.).

Weinberg, Louis. A general RLC synthesis procedure. Proc. I. R. E. **42**, 427–437 (1954).

Consideration is first given to a network transfer ratio K, the ratio of output to input voltages. Then $K=p(s)/q(s)$ where s is the complex frequency and p and q are polynomials with real coefficients. For passive networks q is a Hurwitz polynomial, that is, the zeros are confined to the left half-plane. The problem solved is the determination of a synthetic network with transfer ratio $K_0=p/Hq$ for some constant H. The author writes $q=q_1+Aq_1'$ where q_1 is a Hurwitz polynomial if A is a sufficiently small positive constant. Thus

$$(*)\qquad K_0=(p/q_1)(H+HAq_1'/q_1)^{-1}.$$

The synthetic network is taken to be of the form of a symmetrical Wheatstone bridge. Thus if Z_a and Z_b are the impedances of the bridge arms then

$$(**)\qquad K_0=(Z_b-Z_a)(Z_b+Z_a)^{-1}.$$

The numerators and the denominators in (*) and in (**) are equated. This gives a partial fraction expansion for Z_a and Z_b. If H is sufficiently large, it is seen that Z_a and Z_b may be realized by simple series parallel combination of resistors, inductors, and capacitors. This synthesis method has the desirable features that mutual inductance is not needed and that each inductor appears in conjunction with a series resistor. It also results that capacitors appear in conjunction with shunt resistors. Various modifications are discussed which indicate the flexibility of the method including transfer admittance, transfer impedance, and prescribed output termination. It is shown that a transfer admittance can be realized by an unbalanced network if real transformers are used. *R. J. Duffin.*

Quantum Mechanics

✶Einstein, A. Elementare Überlegungen zur Interpretation der Grundlagen der Quanten-Mechanik. Scientific papers presented to Max Born, pp. 33–40. Hafner Publishing Co. Inc., New York, N. Y., 1953. $2.50.

It is asked whether the present interpretation of quantum mechanics is valid and whether in the classical limit it implies the classical interpretation. If one considers a small ball moving between impenetrable and perfectly reflecting smooth walls a meter apart, it is argued that the classical and quantum interpretations of this physical system are very different. The causal interpretation of quantum theory due to de Broglie and Bohm is also criticized, for according to this the velocity of the ball is zero. *H. C. Corben.*

✶Bohm, D. A discussion of certain remarks by Einstein on Born's probability interpretation of the ψ-function. Scientific papers presented to Max Born, pp. 13–19. Hafner Publishing Co. Inc., New York, N. Y., 1953. $2.50.

The criticism by Einstein [see the preceding review] of the causal interpretation of quantum theory is shown not to apply to the actual experimental situation related to the example cited. Even in macroscopic systems, quantum theory is not required to become identical with classical theory any more than wave theory is required to be identical with geometrical optics. *H. C. Corben.*

✶de Broglie, Louis. L'interprétation de la mécanique ondulatoire à l'aide d'ondes à régions singulières. Scientific papers presented to Max Born, pp. 21–28. Hafner Publishing Co. Inc., New York, N. Y., 1953. $2.50.

A description of the author's attempts to re-interpret quantum theory. Bohm's more recent work allows some of the objections to his pilot wave theory to be answered. An answer to Einstein's criticism [see the paper reviewed second above] is given by noting that the elementary standing-wave picture is not valid unless the de Broglie wave length of the ball exceeds the uncertainty in the position of the wall. *H. C. Corben* (Pittsburgh, Pa.).

✶Schrödinger, Erwin. The general theory of relativity and wave mechanics. Scientific papers presented to Max Born, pp. 65–74. Hafner Publishing Co. Inc., New York, N. Y., 1953. $2.50.

The author argues that space must be closed and finite in order that the wave equations describing the quantum mechanical behavior of matter have a discrete spectrum. He discusses various formulas relating the radius of a spherical universe, the Compton wave length of a proton and the number of particles present in the universe.
A. H. Taub (Urbana, Ill.).

Schwinger, Julian. A note on the quantum dynamical principle. Philos. Mag. (7) **44**, 1171–1179 (1953).

Schwinger's dynamical principle [Physical Rev. (2) **82**, 914–927 (1951); **91**, 713–728 (1953); these Rev. **13**, 520; **15**, 81] is here applied to quantum mechanical systems, with a finite number of degrees of freedom, and obeying first-

order equations of motion. After giving the detailed development of the theory the author states that Burton and Touschek [Philos. Mag. (7) **44**, 161–168 (1953); these Rev. **14**, 1045], in their criticism of his 1951 work, have given an erroneous application of his principles. *C. Strachan.*

Burton, W. K., and Touschek, B. F. Schwinger's dynamical principle. Philos. Mag. (7) **44**, 1180–1181 (1953).

In replying to the paper reviewed above, the authors say that they objected only to an uncritical application of Schwinger's dynamical principle in a less complete form than it has since been given. An additional comment by Schwinger is included. *C. Strachan* (Aberdeen).

Watanabe, Ichie. On the Heisenberg treatment of the field variables. Progress Theoret. Physics **10**, 371–385 (1953).

Integral equations are proposed for vacuum expectation values of chronologically ordered products of fermion and boson operators in Heisenberg representation. Explicit formulas are given for the case of one fermion and a scalar meson in scalar interaction. From these the S-matrix and the Bethe-Salpeter equation in the "ladder" approximation are derived. *A. J. Coleman* (Toronto, Ont.).

Landsberg, Peter T. A proof of Temple's laws of transition. Ann. Physik (6) **14**, 14–16 (1954).

The statistical development of quantum theory proposed by Temple [The general principles of quantum theory, Methuen, London, 1934] involves an assumption which is here proven from certain postulates. The modified statistical basis given here is also applicable to systems containing a small number of particles. *H. C. Corben.*

Infeld, L., and Plebanski, J. Electrodynamics without potentials. Proc. Roy. Soc. London. Ser. A. **222**, 224–227 (1954).

This is a condensed version of a paper by the same authors [Acta Phys. Polonica **12**, 123–134 (1953); these Rev. **15**, 489]. *A. J. Coleman* (Toronto, Ont.).

Landé, Alfred. Thermodynamische Begründung der Quantenmechanik. Naturwissenschaften **41**, 125–131 (1954).

Dirac, P. A. M. The Lorentz transformation and absolute time. Physica **19**, 888–896 (1953).

A general exposition of the author's reasons for retaining the concept of the aether, and an account of how he proposes using it to obtain relativistically invariant physical theories. The first two sections contain a general discussion of how the aether and absolute time may be reinstated in physics with the aid of the general methodology of quantum mechanics. [The author's thoughts on these general aspects have been amplified in Scientific Monthly **78**, 142–146 (1954).] The manner in which these notions may be used in constructing theories is illustrated in the remaining two sections of the paper by applying them to the problem of the motion of an "irrotational" stream of electrons in an electromagnetic field [cf. Dirac, Proc. Roy. Soc. London. Ser. A. **209**, 291–296 (1951); **212**, 330–339 (1952); these Rev. **13**, 893; **14**, 228]. *H. P. Robertson.*

Kothari, L. S. Riesz potential and the elimination of divergences from quantum electrodynamics. II. Proc. Phys. Soc. Sect. A. **67**, 201–205 (1954).

The author applies his version [same Proc. **67**, 17–24 (1954); these Rev. **15**, 586] of the Riesz analytic continuation method to the calculation of the polarization of the vacuum in quantum electrodynamics, working to second order in e. He finds (i) the finite observable effect (Uehling term) in agreement with the usual methods, (ii) a charge-renormalization term which is finite and depends on the arbitrary parameter l_0 introduced by the author's method, (iii) a photon self-energy term which is finite and independent of l_0. The photon self-energy is identical with a result found by G. Wentzel [Physical Rev. (2) **74**, 1070–1075 (1948)] and is physically inadmissible. It shows that the author's method is not gauge-invariant.

F. J. Dyson (Princeton, N. J.).

Petermann, A. Divergence de la théorie de perturbation. Arch. Sci. Soc. Phys. Hist. Nat. Genève **6**, 5–23 (1953).

This paper comes chronologically before several on the same subject which have been reviewed earlier. The author proves the divergence of the perturbation theory series expansions for the case of a scalar field interacting with scalar "photons" of zero rest-mass. The proof is by direct calculation of the general term of the series. The calculation is elementary but complicated. *F. J. Dyson.*

Petermann, A. Une série divergente en représentation intermédiaire. Helvetica Phys. Acta **26**, 731–742 (1953).

The author considers the theory of a single-scalar field with a self-interaction of the form $\lambda\phi^4$. He investigates the convergence of the perturbation power-series expansions of the field-operators in the "intermediate representation" introduced by the reviewer [Physical Rev. (2) **83**, 608–627 (1951); these Rev. **13**, 608]. He finds that one such power-series is divergent, and that it diverges just as badly as the power-series obtained in the usual interaction representation. Contributions from reducible Feynman graphs are neglected, but it is highly unlikely that these would improve the convergence. The proof is only sketched, since it uses the same complicated algebraic machinery as the author's earlier proof of divergence of the interaction-representation series [see the paper reviewed above]. From these investigations we must conclude that the intermediate representation, which is an ugly monstrosity invented solely in order to make the perturbation series converge, has failed in its purpose and should be consigned to oblivion.

F. J. Dyson (Princeton, N. J.).

Karlson, Erik. On regularization by means of analytic continuation. Ark. Fys. **7**, 221–237 (1953).

The author applies a method of regularization originally suggested by Källén [Ark. Fys. **5**, 130–131 (1952)] to the calculation of the vacuum-polarization in quantum electrodynamics. The aim is to verify that the method makes all spurious gauge-dependent terms vanish automatically. This is verified for terms of order e^2 and e^4. Unfortunately the author has not bothered to compute the remaining finite effects in order e^4, which are of some importance for the comparison of the theory with experiment.

F. J. Dyson (Princeton, N. J.).

Plebański, J. Nonlinear electrodynamics and elementary laws. Bull. Acad. Polon. Sci. Cl. III. **1**, 34–38 (1953).

It is assumed that the field equations of electrodynamics are of the second order in the potentials, gauge invariant, and derivable from a Lagrangian function. The Lagrangian then depends on two invariants, $F=\frac{1}{2}(B^2-E^2)$, and $G=(E\cdot B)^2$: $L=L(F, G^2)$. It is shown that if the "elementary law" of force between two static charges is assumed to be

known, then the function $L(F, 0)$ can be determined. Some features of this force are discussed. *N. Rosen* (Haifa).

Valatin, J. G. On the Dirac-Heisenberg theory of vacuum polarization. Proc. Roy. Soc. London. Ser. A. **222**, 228–239 (1954).

The theory of electrons in a given external field is treated by redefining the current and energy-momentum density matrices, compensating their singularities by subtracting appropriate c-numbers in the manner first suggested by Dirac and Heisenberg. The vacuum expectation value of the current density matrix, defined for two points x', x'', is then well-defined for $x'\neq x''$ and by means of the compensating terms this is made finite in the limit when x' and x'' coincide. *H. C. Corben* (Pittsburgh, Pa.).

Pais, A. Isotopic spin and mass quantization. Physica **19**, 869–887 (1953).

The proliferation of new particles in recent years has suggested to many people [e.g. Heisenberg before 1947] the desirability of a theory containing an eigenvalue problem for the masses of physically occurring particles. The author attempts to obtain such a theory for nucleons and heavier particles by incorporating charge invariance into the wave equation. To do this he associates with each event of space-time a two-dimensional sphere which he calls "ω-space". This provides a representation space for the isotopic spin variables. His wave function has eight components depending on the six space-time and ω variables. He assumes that the wave equation is a simple generalization of Dirac's which gives rise to an infinite number of mass values. The lowest and most stable state has two substates of charge $+e$ and 0 which he identifies with p and n. The next two are identified with V-particles. Though some consequences of his theory are consonant with observation, the author emphasises the tentative nature of this first approach. It is reported that in the discussion following the presentation of this paper at the Kamerlingh Onnes Conference in 1952, Pauli commended the author's attempt to relate the empirically observed conservation of the number of nucleons and the charge independence of nuclear forces to group-theoretical properties of the laws of nature.

A. J. Coleman (Toronto, Ont.).

Rayski, J. Mass quantization and isotopic spin in non local-field theory. Nuovo Cimento (9) **10**, 1729–1735 (1953).

The author proposes a new wave-equation for elementary particles of spin $\frac{1}{2}$, starting from the principle of reciprocity of Max Born [Rev. Modern Physics **21**, 463–473 (1949)]. The resulting model is similar to that of A. Pais [see the preceding review] in allowing the particle to possess internal degrees of freedom which may be identified with "isotopic spin" and "isotopic angular momentum". Following the non-local field theory of H. Yukawa [Physical Rev. (2) **77**, 219–226 (1950); these Rev. **11**, 567], he considers a wave-function $\psi(x, r)$ depending on the external coordinates x and the internal coordinates r. Both x and r are 4-vectors transforming alike under Lorentz transformations. Let α_μ and β_μ be two independent sets of Dirac matrices operating on ψ, so that ψ has 16 components. The wave-equation is taken to be

$$(1)\qquad (\alpha_\mu(\partial/\partial x_\mu)-2\beta_\mu(\partial/\partial r_\mu)+k)\psi=0.$$

Note that this equation is not self-reciprocal in the sense of Born. The internal r-space being non-compact, there is no hope of obtaining from (1) alone a discrete set of eigenstates for a particle at rest. Yukawa [loc. cit.] restricted the r-space by imposing on ψ the supplementary conditions

$$(2)\qquad r_\mu(\partial/\partial x_\mu)\psi(x, r)=0,$$

$$(3)\qquad (r_\mu^2-\lambda^2)\psi(x, r)=0.$$

Here (2) is self-reciprocal but (3) is not. So the author imposes instead of (3) the self-reciprocal condition

$$(4)\qquad (\lambda^2(\partial^2/\partial x_\mu^2)-\lambda^{-2}r_\mu^2)\psi=0,$$

where λ is the "universal length" of the theory. For a particle at rest with mass M, (2) and (4) give

$$(5)\qquad r_4\psi=0,\quad (|r^2|-\lambda^4M^2)\psi=0.$$

Thus the r-space is effectively reduced to the surface of a 3-dimensional sphere with radius λ^2M, and discrete eigenstates will be obtained exactly as in the Pais model.

Let τ be Pauli spin matrices in the space of the β-indices, and let l be the orbital angular momentum operator in the r-space. For a particle at rest with mass M, (1) reduces to

$$(6)\qquad (-\alpha_4M+2i\epsilon\beta_4(\lambda^2M)^{-1}(\tau\cdot l+1)+k)\psi=0.$$

Here α_4 and $i\epsilon\beta_4$ have independently the eigenvalues ±1, and $(\tau\cdot l+1)$ has eigenvalues $\pm(j+\frac{1}{2})$ with $j=\frac{1}{2}, \frac{3}{2}, \cdots$. So the eigenvalues of M are given by

$$(7)\qquad M=\tfrac{1}{2}k|1\pm(1\pm8(j+\tfrac{1}{2})(k\lambda)^{-2})^{1/2}|.$$

If the Yukawa condition (3) had been used instead of (4), the mass-spectrum would have been

$$(8)\qquad M=k|1\pm2(k\lambda)^{-1}(j+\tfrac{1}{2})|,$$

identical with that of Pais.

In the reviewer's opinion this model differs essentially from that of Pais in two ways, the difference between (3) and (4) being a matter of detail and at this stage unimportant. First, there is the fact that ψ has 16 components instead of 8. This results in the double ambiguity of sign in the eigenvalues of $i\epsilon\beta_4$ and $(\tau\cdot l+1)$ in (6). There are thus two-independent eigenstates for each choice of signs in (7) or in (8). Such an additional degeneracy is physically inadmissible; for example, if the $j=\frac{1}{2}$, $l=0$ solution is to be identified with the proton, then there would exist two states of a proton with given spin and charge, in contradiction to the known statistical behaviour of protons. Second, there is the fact that the internal r-space transforms with the external x-space under rotations of the coordinate system. This would seem to imply that the isotopic spin should be included as part of the ordinary angular momentum of the particle, which is again contrary to all experience. The reviewer believes that the second objection to the author's model can be removed only by detaching completely the internal space from Lorentz transformations of the external space. If this is done, and some extra supplementary condition is imposed to remove the first objection, then the model becomes only a variant of the Pais model, and all connexion with the Born-Yukawa space-time structure is lost.

F. J. Dyson (Princeton, N. J.).

Darling, B. T. Field theory of equations with many masses. Physical Rev. (2) **92**, 1547–1553 (1953).

In an earlier paper [Physical Rev. (2) **80**, 460–466 (1950); these Rev. **12**, 465] the author derived an integro-difference equation for spin-$\frac{1}{2}$ material processes, arising out of his theory of "irreducible volume character of events". He was then able to show that this integro-difference equation is equivalent to a partial differential equation of infinite order, $D(z)\psi=0$, where $D(z)=[-2J_2(z)+(1.6e^2/\hbar c)J_1(z)]/z$. Here

J_1 and J_2 are Bessel's functions, and $z=-2w\gamma^\lambda\partial/\partial x_\lambda$, w being a constant with the dimensions of length.

In this paper the interaction of spin-½ field, described by the equation above, with the electro-magnetic field is studied. The interaction is introduced by replacing z by the gauge-invariant quantity $Z=z+ieA\lambda/hc$. An unpublished theorem of Leichter is used, which states that the general solution of $D(Z)\psi=0$ is a superposition of non-orthogonal mass states, designated as "root fields". It is shown that general expressions for the current-vector and the energy-momentum tensor exist and these decompose into a sum over those of individual mass states, but with an alternation of sign for consecutive roots. The Lagrangian decomposes into an alternating sum over individual free Lagrangians plus the usual term $c^{-1}j_\mu A_\mu$. The matter field is quantised by treating the root fields as independent anti-commuting fields. The results of the paper would contradict the remarks of Pais and Uhlenbeck [ibid. **79**, 145–165 (1950); these Rev. **12**, 227] about quantisation of such a multiple-mass Dirac field in the presence of the electro-magnetic field. *A. Salam* (Cambridge, England).

Tanaka, Shô, and Umezawa, Hiroomi. On the transition matrix and the Green function in the quantum field theory. Progress Theoret. Physics **10**, 617–629 (1953).

The authors show that the many-body kernels defined by Schwinger [Proc. Nat. Acad. Sci. U. S. A. **37**, 452–455, 455–459 (1951); these Rev. **13**, 520] contain information about transition amplitudes of the states. By explicit calculation, it is shown that many-body kernels for complicated processes can be expressed in terms of one-body kernels G, $\mathfrak{G}$ and the vertex operator Γ, to any desired order of approximation. *A. Salam* (Cambridge, England).

Power, E. A. A new proof of the perturbation expansions in quantum mechanics. Proc. Roy. Soc. London. Ser. A. **218**, 384–391 (1953).

An algebraical formalism for the derivation of the perturbation formulas of quantum mechanics is developed where the non-commutative character of the perturbation operators is taken into account explicitly. No recourse is made to the series of eigenfunctions of the free system: "an advantage similar to the use of vectors as compared with Cartesian methods in classical mechanics". The formalism is first applied to time-independent, and then to time-dependent perturbations. The formulas obtained resemble the well-known perturbation formulas of quantum mechanics, but are somewhat more general. Terms arising from intermediate states with and without energy conservation can be easily distinguished. The formalism is set up in the Schrödinger representation. The formulas obtained are shown to be equivalent to formulas derived in the interaction representation. *E. Gora* (Providence, R. I.).

Schönberg, M. A general theory of the second quantization methods. Nuovo Cimento (9) **10**, 697–744 (1953).

The methods of second quantization are applied to a general linear equation, of first order in the time derivative, describing the time evolution of an n-particle wave function in terms of one, two, three, . . . particle energy operators. Both the proofs, here given in full detail, and the formalism obtained are essentially identical with the familiar contents of the quantum-mechanical second quantization method. The formalism is applied to the following equations: the Schrödinger equation of a quantum-mechanical system, leading to a statistical theory for an assembly of non-interacting replicas of the system (essentially equivalent to the use of von Neumann's density matrix); the Schrödinger equation for a system of identical particles in interaction with each other, leading to ordinary second quantization; the Liouville equation of classical mechanics for a system of interacting particles, leading to a formalism proposed earlier by the author [Nuovo Cimento (9) **9**, 1139–1182 (1952); **10**, 419–472 (1953); these Rev. **14**, 710]. The paper ends with the derivation of ergodic theorems of a type similar to those usually derived in operator theory: the existence of a generalized Cesàro or Abel limit is shown for an operator $\exp(-iKt)A\exp(iKt)$ when $t\to\infty$. *L. Van Hove* (Princeton, N. J.).

Visconti, A. Sur quelques applications du formalisme de l'opérateur d'évolution. J. Phys. Radium (8) **14**, 591–603 (1953).

By a systematic treatment of the integral equations defining the unitary operator $U(t,t')$ which describes the development of a quantum-mechanical system, the author gives a neat derivation of the basic equations of recent theories of Stueckelberg-Tomanaga-Schwinger, Feynman, Heitler and Arnous-Zienau. The approach seems more elegant than those previously available and puts in evidence interrelations among these theories. Of particular interest is a variational method for the simultaneous approximation to $U(t,t')$ and the associated resolvent operator. *A. J. Coleman* (Toronto, Ont.).

Fabre de la Ripelle, Michel. Méthode de résolution des équations de perturbation pour un hamiltonien de perturbation quelconque. C. R. Acad. Sci. Paris **238**, 1291–1293 (1954).

One defines $g_n(m_{i+1},m_i)$ as the ensemble of graphs leading from the state m_i to the state m_{i+1} after going through $(n-1)$ intermediate states without any of these being either m_i or m_{i+1}. The ensemble of all graphs from m_a to m_b such that $g_n(m_b,m_a)$ is the first non-zero graph is expressed in terms of $\sum_n g_n$. *H. C. Corben* (Pittsburgh, Pa.).

Chenon, René. Nouvelle présentation de la théorie covariante des champs. C. R. Acad. Sci. Paris **238**, 1382–1384 (1954).

The relation between the Heisenberg and Schrödinger representations is developed covariantly without introducing space-like surfaces, by considering the Hermitean four-vector operator $ih\partial_\mu U\cdot U^{-1}$, where U is the unitary operator relating the two representations. *H. C. Corben.*

Buneman, O. Self-consistent electrodynamics. Proc. Cambridge Philos. Soc. **50**, 77–97 (1954).

The potentials of Dirac's "new electrodynamics" are here identified with the retarded potentials created by the streaming charge. It is shown that the conservation of circulation of the generalized momentum vector implies motion according to the Lorentz force law. The converse had been proved previously but the present result makes possible a neat enunciation of the new theory.

Most of the paper is devoted to the discussion of stationary self-consistent flow patterns, including (i) a case of constant density similar to one occuring in London's theory of superconductivity, (ii) an infinite helical flow, (iii) a flow with cylindrical symmetry which could maintain itself. Self-maintaining flows such as (iii) are of special interest as possibly providing a model for a finite electron with spin and magnetic moment. The only self-maintaining pattern

so far discovered is, unfortunately, infinite in extent. The author hopes that a finite self-maintaining flow would provide an explanation of the fine-structure constant. The concluding sentence of the paper contains the undoubted truth: "An element of optimism is certainly needed in this search for an electron model." *A. J. Coleman.*

Corinaldesi, E. On the scattering theory of relativistic equations. Nuovo Cimento (9) 10, 1673–1680 (1953).

Extends the investigation of Parzen [Physical Rev. (2) 80, 261–268, 355–360 (1950); 81, 808–814 (1951); these Rev. 12, 571] on potential scattering at high energies governed by the Klein-Gordon or Dirac equations to include potentials with a pole at the origin. An explicit formula, analogous to Parzen's, is obtained for the limit of the phase shift at infinite energy. *A. J. Coleman* (Toronto, Ont.).

Dyson, F. J. The wave function of a relativistic system Physical Rev. (2) 91, 1543–1550 (1953).

Two approaches have, in the past, been employed for handling relativistic systems: the 3-dimensional approach of Tamm [Acad. Sci. USSR J. Phys. 9, 449–460 (1945)] and Dancoff [Physical Rev. (2) 78, 382–385 (1950)] and the 4-dimensional approach of Salpeter and Bethe [ibid. 84, 1232–1242 (1951); these Rev. 14, 707]. The wave function in the Tamm-Dancoff approach is represented by a set of probability amplitudes $\alpha(N)$ for finding a set $N=(N_1, N_2, \cdots)$ of free particles in each of the normal modes of the non-interacting fields. (More precisely, $\alpha(N)=(N_1!N_2!\cdots)^{-1/2}(\Phi_0{}^*A(N)\Psi)$, where $A(N)$ is the product of annihilation operators for particles specified by N, Ψ is the constant state vector of the system and Φ_0 the state vector for the vacuum of the non-interacting fields.) In the Bethe-Salpeter approach, the wave function is formally defined as a matrix element of a particular product of Heisenberg operators, taken between the vectors Ψ and Ψ_0, the latter being the state vector for the vacuum of the interacting fields. The 3-dimensional wave function has a clear physical meaning, but it runs into serious difficulties connected with vacuum self-energy, while the physical interpretation of the 4-dimensional wave function, notwithstanding its other advantages, has remained unclear. Nor had an exact correlation been, so far, found between the two approaches.

In this paper the author makes an important advance in finding such a correlation. A new type of amplitude $a(N, N')$ is introduced, defined as

$$a(N, N')=(N_1!N_2!\cdots)^{-1/2}(N_1'!N_2'!\cdots)^{-1/2} \times(\Psi_0{}^*C(N)A(N')\Psi).$$

$C(N)$ is the product of creation operators for particles N; $a(N, N')$ thus describes "probability in the state Ψ of finding N' particles more, and N particles less, than are to be found in the vacuum of the interacting fields".

In an earlier letter [ibid. 90, 994 (1953)] where these amplitudes are first defined the author showed that by correctly and automatically incorporating the fact of Ψ_0 being the comparison state for the system, these amplitudes remove all difficulties connected with the vacuum self-energy. The results of this paper are: (1) If in the Bethe-Salpeter wave function, the operators concerned are defined on the same time-plane, the resulting wave function can be expressed as a linear combination of a subset of $a(N, N')$'s. (2) If $\beta(N)$ denotes the Tamm-Dancoff amplitudes for the vacuum state Ψ_0, $a(N, N')$ can be written as a linear sum of the products of $\alpha(N)$'s and $\beta^*(N)$'s. (3) A sufficient condition for $\sum_N\sum_{N'}|a(N, N')|^2$ to be finite is that $\sum_N 2^N|\alpha(N)|^2<\infty$ and $\sum 2^N|\beta(N)|^2<\infty$. "In practice it is very likely that these (latter) conditions would be fulfilled in cases where Tamm-Dancoff point of view is applicable." (4) The Tamm-Dancoff formalism with a cut-off (i.e., setting $\alpha(N)$'s and $\beta(N)$'s involving more than say Q particles equal to zero; this is necessarily required in practice) is equivalent to a theory in which the cut-off is directly applied to the amplitudes $a(N, N')$.

Finally, as an illustration of the method, an integral equation is set up for the proton-neutron system by taking all amplitudes $a(N, N')$ involving four or more particles equal to zero. The section concludes with a clear discussion of the role of the boundary condition that Ψ_0 is the vacuum state in restricting the solutions of the integral equation.

A. Salam (Cambridge, England).

Schweber, S. S. Covariant formulation of the Tamm-Dancoff method. Physical Rev. (2) 94, 1089 (1954).

Cini, M. A covariant formulation of the non-adiabatic method for the relativistic two-body problem. I. Nuovo Cimento (9) 10, 526–539 (1953).

A covariant formulation of the Tamm-Dancoff non-adiabatic treatment of two-particle interaction is given which permits a determination of both the contributions from convergent processes and from non-convergent radiative processes in a consistent way. The interaction representation of quantum field theory is used. A neutral pseudoscalar field is assumed. For a function $A_\sigma(x', x'')$, which represents the probability amplitude of the state vector describing a free proton and a free neutron at two different space-time points x', x'' respectively, an integral equation is deduced where all the terms up to the fourth order in the coupling constant G are retained. In the G^2-approximation this equation reduces to the Tamm-Dancoff integral equation. The method is formulated in such a way that it is suitable for the treatment of bound states only. It is more involved than the Salpeter-Bethe method [Physical Rev. (2) 84, 1232–1242 (1951)], but the meaning of the quantities used in the calculations can be more easily recognized.

E. Gora (Providence, R. I.).

Cini, M. A covariant formulation of the non-adiabatic method for the relativistic two-body problem. II. Nuovo Cimento (9) 10, 614–629 (1953).

The integral equation obtained in part I [see the preceding review] involves infinite kernels corresponding to radiative processes. The techniques of Dyson [Physical Rev. (2) 75, 486–502, 1736–1755 (1949); these Rev. 10, 418; 11, 145] and others for the renormalization of the S-matrix are used to separate from these kernels convergent parts. The renormalization of the equation in the G^2-approximation is carried out first. The renormalization of the divergent kernels of "exchange" type (kernels describing the exchange between proton and neutron of at least one meson) is worked out in detail. To renormalize other kernels containing reducible graphs Dyson's methods are used in a straightforward way. A brief discussion of the divergent kernels which describe disconnected loops in the vacuum leads to the conclusion that such kernels cannot be renormalized; they can, however, be eliminated if the same approximation is used as in the S-matrix theory, namely that the time dependence of the state vector can be neglected.

E. Gora (Providence, R. I.).

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